Gain of Phased Array Antennas Under Small Random Errors in Element Placement

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Dedications

To my family,

I am truly blessed to have your love and support.
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Abstract
Gain of Phased Array Antennas Under Small Random Errors in Element Placement

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A phased array is an arrangement of antennas whose emergent radiation pattern is controlled by the relative phases of the signals that are feeding its elements. Within implementation limits, the effective radiation pattern of the array is reinforced in a certain desired direction and repressed in other undesired directions. The pattern can thus be steered by introducing a relative phase shift between its elements so that its radiation adds constructively in a certain direction. This mechanism also means that error in the relative spacing between the elements of an array is likely to result in an error in the phase and a decrease in the gain of the array in the desired direction.

The effects of element spacing error in phased array systems have been studied extensively. The approach in most studies was to show first that errors in the relative spacing of the elements results in error in the relative phase between the elements, then to characterize the distribution of the resulting gain. Often it was assumed that there are many elements in the array, and mathematical limits were used to calculate the radiation pattern under a large array assumption.

Like most studies of the subject, we first quantize how errors in the relative spacing of the elements affect the error in the relative phase between the elements. We use a small-error assumption that is realistic in planned deterministic phased arrays. However, we do not make the assumption of a large number of elements in the array and our results are applicable to small arrays as well. We show that the gain loss can be approximated by the sum of the squares of the relative phase of each element with a scaling factor. Thus whenever the relative phase of each element is normally distributed with zero mean, the gain loss is distributed as a gamma distribution.

Our key result is an expression for the allowed variance in the element position as a function of gain loss, the number of elements of the array, and the probability that this gain loss is realized. Using the expression a designer can specify placement tolerance as a function of the quality (in terms of gain in the desired direction) of the array. We use the expression for the element position variance to study the design of uniform linear arrays, uniform rectangular arrays, and uniform circular arrays. Simulations demonstrate that our derived distribution for the gain loss is very close to the expected distribution.
1. Introduction

1.1 Overview

A phased array is an arrangement of antennas whose emergent radiation pattern is controlled by the relative phases of the signals that feed its elements. Within implementation limits, the effective radiation pattern of the array is reinforced in a certain desired direction and repressed in other undesired directions. The pattern can thus be steered by introducing a relative phase shift between its elements so that its radiation adds constructively in a certain direction. This mechanism also means that error in the relative spacing between the elements of an array is likely to result in an error in the phase, and a decrease in the gain of the array in the desired direction.

The effects of element spacing error in phased array systems have been studied extensively [2, 7, 8, 17]. The approach in most of these studies was to show first that errors in the relative spacing of the elements translate into errors in the relative phase between the elements, then to characterize the distribution of the resulting gain. Most studies on the subject have assumed that there are many elements in the array and have used mathematical limits under a large array assumption.

1.2 Assumptions

In this study, we consider radiation patterns that are far from their source so that we can make use of the far-field radiation approximation. We ignore the effects of mutual coupling between elements on the radiation pattern. When speaking of the array gain, we assume that each element of the array is an ideal isotropic radiator whose relative feed coefficients have equal magnitude. We also use the term gain synonymously with directive gain and directivity. Lastly, we only consider error in the relative displacement between the elements that is normally distributed with zero mean and small variance.

1.3 Statement of the Research Problem

We calculate the effects of random errors in the element spacing on the performance of phased antenna arrays. We aim to characterize the distribution of the resulting gain loss of such errors, particularly for arrays with small numbers of elements. With this distribution, we aim to find an expression for the allowed variance in the element position as a function of gain loss, the number of
elements of the array, and the probability that this gain loss is realized.

1.4 Organization of this Thesis

Chapter 2 summarizes the background information on phased array systems needed in order to understand the rest of the study. Specifically, we derive the radiation vector of an antenna under the assumption of the far-field approximation. We then define the gain of an antenna as well as the array factor. Next, we describe the mechanism by which phased array radiation patterns are steered. We conclude the background section by describing the array geometries considered in this study.

In chapter 3, we derive and characterize two different estimates for the gain loss distribution of an array with normally distributed phase errors – one of these expressions uses the central limit theorem as is common in the literature, the other does not. To this end, we derive a useful expression for the gain loss of the array in terms of the antenna phase errors. From this expression, we show that the gain loss is distributed as a gamma distribution. We also follow [2] to derive an alternative estimate of the gain loss distribution – this estimate of the gain loss is a scaled non-central Chi-squared distribution. We conclude this chapter by calculating the mean and variance of each of these distributions.

We begin chapter 4 by providing an expression for the phase variance as a function of the gain loss, the number of elements in the array, and the probability that this gain loss is realized. We calculate two such expressions for the variance, one based on the assumption that the gain loss is a gamma distribution, the other based on the assumption that the gain loss is a scaled non-central Chi-squared distribution. We apply the phase variance expression to an array whose elements have normally distributed relative spacing errors. We show that the normally distributed relative spacing errors will lead to a normally distributed relative phase errors. By relating the variance of the phase to the variance of the position, we derive an expression for the allowed position variance as a function of the gain loss, the number of elements in the array, and the probability that this gain loss is realized.

In chapter 5 we assess the accuracy of our derived expressions through simulations. This assessment is of particular importance in the context of our “small error” assumptions – we want to know at what size of error parameters our expressions cease to be accurate. We analyze our estimated distribution of the gain loss by qualitatively and quantitively comparing it to the expected distribu-
tion obtained from Monte Carlo simulations. Next, we compare the two gain loss distributions that we have derived. We are particularly interested in comparing how well each of these distributions approximates the simulated distribution as a function of the number of array elements. Finally, we quantitatively analyze our expression for the allowed position variance by means of Monte Carlo simulations.

Chapter 6 provides a summary of our findings.
2. Phased Array Antennas

We introduce some of the fundamental concepts behind the operation of phased array antenna systems. In the first section, we derive the radiation vector of an antenna element under the far-field approximation. We show that the radiation vector represents the radiation pattern of an antenna and that it is the 3-dimensional Fourier transform of the current density of the source. Next, we discuss the principle of operation of phased arrays and derive an important quantity known as the array factor. We then discuss the array’s gain and show that for phased arrays it is closely related to the array factor. We show how phased array antenna systems can be electrically steered by adding relative phase shifts between the elements of the array. Finally, we discuss three important array geometries considered in this study: the uniform linear array, the uniform rectangular array, and the uniform circular array.

2.1 The Radiation Pattern at the Far-field

It is well known that the electric and magnetic fields generated by charge and current distributions are described by Maxwell’s equations. In differential form, these are given by

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}; \tag{2.1}
\]

\[
\nabla \cdot \mathbf{B} = 0; \tag{2.2}
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \tag{2.3}
\]

\[
\nabla \times \mathbf{B} = \mu \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}; \tag{2.4}
\]

where \(\mathbf{E}\) is the electric field, \(\mathbf{B}\) is the magnetic field, \(\rho\) is the charge density, \(\mathbf{J}\) is the current density, \(\epsilon\) is the electric permittivity, \(\mu\) is the magnetic permeability, and \(c\) is the speed of light. It can be shown [14] that equations (2.2) and (2.3) imply the existence of an electric scalar potential \(\psi(\mathbf{r}, t)\) and a magnetic vector potential \(\mathbf{A}(\mathbf{r}, t)\) that satisfy

\[
\mathbf{E} = -\nabla \psi - \frac{\partial \mathbf{A}}{\partial t}; \text{ and} \tag{2.5}
\]

\[
\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.6}
\]
The electric and magnetic potential functions defined above are not unique; therefore, there is a fairly large amount of freedom in defining these functions. Often we apply a constraint known as the *Lorenz gauge* which is given by
\[
\nabla \cdot A + \frac{1}{c^2} \frac{\partial \psi}{\partial t} = 0. \tag{2.7}
\]
This condition allows us to greatly simplify the differential equations (2.1–2.4).

By substituting (2.5) into (2.1) and (2.6) into (2.4) and applying the Lorenz gauge, we get
\[
\nabla \cdot (-\nabla \psi - \frac{\partial A}{\partial t}) = -\nabla^2 \psi - \frac{\partial}{\partial t} (\nabla A) = -\nabla^2 \psi - \frac{\partial}{\partial t} \left(-\frac{1}{c^2} \frac{\partial \psi}{\partial t}\right) = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = \frac{\rho}{\epsilon}; \text{ and} \tag{2.8}
\]
\[
\nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} = \nabla \times (\nabla \times A) - \frac{1}{c^2} \frac{\partial}{\partial t} (-\nabla \psi - \frac{\partial A}{\partial t}) = \nabla \times (\nabla \times A) - \nabla (\nabla \cdot A) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu J; \tag{2.9}
\]
where in (2.9) we used the identity \(\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A\).

Equations (2.8) and (2.9) show that Maxwell’s equations take the form of wave equations for the potentials \(A\) and \(\psi\) with sources \(\rho\) and \(J\). In the context of both radar and communications, we are typically interested in fields that have radiated far from their current source. In order to detect an object at great distance and determine useful information about it such as its range and speed, or to transmit information toward a remote receiver, it is important that our antennas are able to adequately transmit power in the desired direction. It is also necessary that we have a good characterization of how power is being radiated in other directions as well. As we show next, this information can be obtained by examining the magnetic vector potential function at far distances.

When the current density of the source is known, the solution of (2.9) takes the form of what is known as the *retarded potential*
\[
A(\mathbf{r}, t) = \int_V \frac{\mu J(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi |\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'. \tag{2.10}
\]
Here \(\mathbf{r}\) is the field (observation) point, \(\mathbf{r}'\) is the source point, and \(V\) is the volume in which the current density of the source resides. From (2.10) we see that the magnetic potential at an observation point
\( \mathbf{r} \) is the sum of the potentials brought about by the infinitesimal current densities within the source at a time \(|\mathbf{r} - \mathbf{r}'|/c\) seconds prior to \(t\). If we assume that \(J\) and \(A\) are each sinusoidal with frequency \(\omega\), we can write them as

\[
J(\mathbf{r}, t) = J(\mathbf{r}) e^{j\omega t}, \quad \text{and} \quad A(\mathbf{r}, t) = A(\mathbf{r}) e^{j\omega t}.
\]

(2.11)

Substituting these into the equation for the magnetic potential, cancelling \(e^{j\omega t}\) from both sides, and letting \(k = \omega/c\) results in

\[
A(\mathbf{r}) = \frac{\mu J(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \int_V J(\mathbf{r}') e^{-jkr} d^3 \mathbf{r}'.
\]

(2.12)

Furthermore, if we assume that the field point is far from the current source (i.e. \(r \gg r'\)) then \(|\mathbf{r} - \mathbf{r}'| \simeq \mathbf{r} - \mathbf{r}'\), where \(\hat{\mathbf{r}}\) is a unit vector pointing towards the far-field observation point – this is known as the far-field approximation. Hence the magnetic potential reduces to

\[
A(\mathbf{r}) = \frac{\mu e^{-jkr}}{4\pi r} \int_V J(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} d^3 \mathbf{r}'.
\]

(2.13)

If we now define the radiation vector as

\[
F(\hat{\mathbf{r}}) = F(\theta, \phi) = \int_V J(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} d^3 \mathbf{r}',
\]

(2.14)

then the magnetic potential is calculated as

\[
A(\mathbf{r}) = \frac{\mu e^{-jkr}}{4\pi r} F(\theta, \phi).
\]

(2.15)

On a sphere of constant radius the \(\frac{\mu e^{-jkr}}{4\pi r}\) factor in (2.15) is merely a scaling factor. Hence, as a function of direction, the radiation pattern of the antenna is described by the radiation vector. In equation (2.14), the radiation vector is the 3-dimensional Fourier transform of the current density; therefore, as we will show in section 2.3, a translation of an antenna in the space domain will result in a phase shift in the frequency domain.
2.2 Antenna Gain

In general, the gain of an antenna in the direction of \( \hat{r} \) is defined as the intensity of the antenna in the direction \( \hat{r} \) divided by a reference power. There are a few different types of gain, each of which are calculated similarly but with a different reference power [7]. It can be shown [14] that the intensity of radiation in the direction of \( \hat{r} \) (in watts per steradian) is given by \( I(\hat{r}) = |A(\hat{r})|^2 \). Therefore, the gain of an antenna is calculated as

\[
G(\hat{r}) = \frac{4\pi (\text{intensity in direction } \hat{r})}{\text{reference power}} = \frac{4\pi |A(\hat{r})|^2}{P_{\text{ref}}} = \frac{4\pi I(\hat{r})}{P_{\text{ref}}}. \tag{2.16}
\]

Three gains commonly considered in radar studies are the realized gain, the power gain, and the directive gain. The realized gain is calculated in (2.16) with the reference power equal to the total power incident at the antenna. The power gain corresponds to a reference power equal to the total power accepted by the antenna. The directive gain corresponds to a reference power equal to the total power radiated by the antenna. In this study we are mainly concerned with the directive properties of antennas; therefore, we will use the directive gain.

It is also common to describe the gain of an antenna as a ratio of its radiation to that of an ideal isotropic radiator. An isotropic radiator is a hypothetical lossless antenna having equal radiation intensity in all directions. Therefore, its radiation intensity in any direction is given by \( I_{\text{iso}} = P_{\text{ref}}/4\pi \). By this definition, the gain of an antenna is

\[
G(\hat{r}) = \frac{4\pi I(\hat{r})}{P_{\text{ref}}} = \frac{I(\hat{r})}{I_{\text{iso}}}. \tag{2.17}
\]

2.3 Array Factor

We consider an array of antenna elements (each of which is an electromagnetic source) spread about a reference origin. The \( N \) identical elements of the array are located at positions \( \mathbf{d}_1, ..., \mathbf{d}_N \), as shown in figure 2.1. Suppose that the antenna elements are fed with sinusoidal signals weighted by complex feed coefficients \( a_1, ..., a_N \); then the current density of element \( n \in [1, ..., N] \) is given by \( J_n(\mathbf{r}) = a_n J(\mathbf{r} - \mathbf{d}_n) \), where \( J(\mathbf{r}) \) is the current density of an antenna element located at the origin with a feed coefficient of one. From equation (2.14), we can calculate the radiation vector of element
\[ F_n(\hat{r}) = \int_V J_n(r)e^{j\hat{r} \cdot r} d^3r = \int_V a_n J(r - d_n)e^{j\hat{r} \cdot r} d^3r = a_n \int_V J'(r')e^{j\hat{r} \cdot (r' + d_n)} d^3r' = a_n e^{j\hat{r} \cdot d_n} F_e(\hat{r}), \quad (2.18) \]

where \( F_e(\hat{r}) \) is the radiation vector of an identical antenna element located at the origin. As we saw in the previous section, the radiation vector was the superposition of the radiation from the infinitesimal current densities within the source. Therefore, the total radiation vector of an array is given by the sum of the radiation vectors of all the elements of the array. If we define a quantity known as the array factor as

\[ F_A(\hat{r}) = a_1 e^{j\hat{r} \cdot d_1} + a_2 e^{j\hat{r} \cdot d_2} + ... + a_N e^{j\hat{r} \cdot d_N}, \quad (2.19) \]

then the total radiation vector of the array is given by

\[ F_{total}(\hat{r}) = F_1(\hat{r}) + F_2(\hat{r}) + ... + F_N(\hat{r}) = a_1 e^{j\hat{r} \cdot d_1} F_e(\hat{r}) + a_2 e^{j\hat{r} \cdot d_2} F_e(\hat{r}) + ... + a_N e^{j\hat{r} \cdot d_N} F_e(\hat{r}) = F_A(\hat{r}) F_e(\hat{r}). \quad (2.20) \]

The radiation pattern of an array of identical antennas is the product of the array factor and the element pattern.
In this study, we assume that the individual antenna elements are ideal isotropic radiators and hence (from equation (2.17)) their individual directive gains are equal to one. From equation (2.19), the array factor has a maximum equal to the number of elements in the array. Therefore, under these assumptions, the array gain is the square of the magnitude of the array factor divided by the number of elements squared. The resulting expression for the gain is

\[ G_A(\hat{r}) = \left| F_A(\hat{r}) \right|^2 / N^2. \]  

\((2.21)\)

### 2.4 Steering a Phased Array

In equation (2.19), the array factor is the sum of \( N \) complex numbers. It therefore follows from a geometric argument that the array factor is maximum whenever of all of these complex numbers have the same phase. Without loss of generality, suppose the phase of the first element is equal to zero in some direction \( \hat{r}_0 \). Then in order for the radiation to be maximum in the direction of \( \hat{r}_0 \), the phase of each element must be equal to zero. This can be achieved by setting the feed coefficients to

\[ a_n = e^{-jk\hat{r}_0 \cdot d_n}, \]  

\((2.22)\)

so that the array factor is given by

\[ F_A(\hat{r}) = e^{jk(\hat{r} \cdot d_1 - \hat{r}_0 \cdot d_1)} + e^{jk(\hat{r} \cdot d_2 - \hat{r}_0 \cdot d_2)} + \ldots + e^{jk(\hat{r} \cdot d_N - \hat{r}_0 \cdot d_N)}. \]  

\((2.23)\)

Now, whenever \( \hat{r} = \hat{r}_0 \), the array factor is equal to \( N \) and hence the normalized gain is equal to one. The gain reaches its maximum value in the direction of \( \hat{r}_0 \). This resulting method of steering an array is often referred to as **beam cophasal excitation** since the radiation from each element in the array is in phase in the direction of \( \hat{r}_0 \). In this case, the phase of each element is set to

\[ \psi_n = -k\hat{r}_0 \cdot d_n. \]  

\((2.24)\)

In order to steer an array in a desired direction, we need to add a phase shift to each element so that their radiation contributions add up in the desired direction. With modern electronics, the speed with which we can adjust the phases of the antenna elements (and thus steer the array) is substantially faster than if we had to rotate the antenna mechanically. Furthermore, a large array can be controlled such that it has multiple beams scanning in different directions.
Steering an array by weighting all of the elements the same and adjusting their relative phase is by no means the most effective method of steering an array. Different distributions of weights of the relative feed coefficients can produce different desired qualities in the radiation pattern such as a wider beamwidth and lower side-lobe ratios. However, these characteristics do typically come at some cost to the maximum gain. In the current study, we restrict our attention to arrays where all of the feed coefficients have the same magnitude.

2.5 Array Geometries Considered in this Study

Perhaps the simplest form of phased array antenna is the uniform linear array (ULA) – an array of antenna elements uniformly spaced along a line. Based on the geometry of figure 2.2(a), we calculate the relative phase of element $n$ of the array from equation (2.24) as

$$\psi_n = -k\langle \sin(\theta_0), \sin(\theta_0) \cos(\pi/2), \cos(\theta_0) \rangle \cdot \langle d(n - 1), 0, 0 \rangle = -kd(n - 1) \sin(\theta_0). \quad (2.25)$$

Expression (2.25) intuitively makes sense; if we imagine a signal arriving at an angle $\theta_0$ in figure 2.2(a), then the difference in path length between adjacent elements for the arriving signal is $d \sin(\theta_0)$. Thus the phase difference of the signals arriving at two adjacent elements is $2\pi$ times the ratio of the difference in path length to the wavelength. This is the expression given by equation (2.25).

A natural extension to the ULA is the uniform rectangular array (URA). A URA is a rectangular grid of antenna elements. It could also be thought of as a series of side-by-side ULAs. From figure
2.2(b) and equation (2.24), we have that the phase of element $n$ of a URA may be calculated as

$$\psi_n = -k \langle \sin(\theta_0) \sin(\phi_0), \sin(\theta_0) \cos(\phi_0), \cos(\theta_0) \rangle \cdot \langle d(n_1 - 1), d(n_2 - 1), 0 \rangle$$

$$= -kd(n_1 - 1) \sin(\theta_0) \sin(\phi_0) - kd(n_2 - 1) \sin(\theta_0) \cos(\phi_0).$$

Unlike the ULA, the URA can be cophasally steered in both directions $\theta$ and $\phi$.

A conformal array is defined as an antenna array that conforms to a surface whose shape is determined by considerations other than electromagnetic [6]. An example of a conformal array would be an array that is integrated onto an aircraft wing. The basic building block for conformal arrays is the uniform circular array (UCA). For example, a cylindrical array is a stack of UCAs. Therefore, in the field of conformal array design, the UCA is very important. For the UCA shown in figure 2.2(c), the relative phase for element $n$ is calculated as

$$\psi_n = -k \langle \sin(\pi/2) \sin(\phi_0), \sin(\pi/2) \cos(\phi_0), \cos(\pi/2) \rangle \cdot \langle \sin(2\pi/n), \cos(2\pi/n), 0 \rangle$$

$$= -kd \sin(2\pi/n) \sin(\phi_0) - kd \cos(2\pi/n) \cos(\phi_0),$$

where $d$ is the radius of the array. We have assumed that the array is only steered along $\phi$ as is common for UCA applications.
3. Gain Loss Distribution

As we saw in chapter 2, the gain of an antenna array is directly related to the relative phase between the signals arriving at the elements of the array. Error in the relative phase of the elements will likely result in reduction in the overall gain of the array. In this chapter, we derive the distribution of the gain loss of the main beam when small phase error is present. We show that when the relative phase of each element is normally distributed about zero with small variance, the gain loss can be very closely approximated by a gamma distribution.

3.1 Expression for the Gain Loss

Suppose we have an \( N \) element array of arbitrarily spaced isotropic elements with random phases \( \Psi = [\Psi_1, ..., \Psi_N]^T \). Then the array factor is given by equation (2.19)

\[
F_A = e^{j\Psi_1} + e^{j\Psi_2} + \ldots + e^{j\Psi_N}.
\]  

(3.1)

The magnitude of \( F_A \) is

\[
|F_A| = \sqrt{\left( \sum_{n=1}^{N} \cos(\Psi_n) \right)^2 + \left( \sum_{n=1}^{N} \sin(\Psi_n) \right)^2}.
\]  

(3.2)

By the multinomial theorem, we write the squared sum terms as

\[
\left( \sum_{n=1}^{N} \cos(\Psi_n) \right)^2 = \sum_{n=1}^{N} \cos^2(\Psi_n) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos(\Psi_i) \cos(\Psi_j), \quad \text{and} \tag{3.3}
\]

\[
\left( \sum_{n=1}^{N} \sin(\Psi_n) \right)^2 = \sum_{n=1}^{N} \sin^2(\Psi_n) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sin(\Psi_i) \sin(\Psi_j). \quad \text{and} \tag{3.4}
\]

By substituting (3.3) and (3.4) into (3.2) and simplifying with trigonometric identities, the array factor magnitude is

\[
|F_A| = \sqrt{N + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos(\Psi_i - \Psi_j)}. \tag{3.5}
\]
Thus from equation (2.21), an expression for the array gain is

\[ G_A = \frac{|F_A|^2}{N^2} = \frac{1}{N} + \frac{2}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos(\Psi_i - \Psi_j). \]  

To further simplify the expression (3.6), we shall assume that the relative phase differences between any two elements in the array is close to zero. This assumption allows us to use the small angle approximation to the cosine. Applying the small angle approximation to the cosine terms in (3.6) in an array gain of

\[ G_A \approx 1 + \frac{2}{N^2} \frac{N(N-1)}{2} - \frac{2}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{(\Psi_i - \Psi_j)^2}{2} = 1 - \frac{1}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\Psi_i - \Psi_j)^2. \]  

The array gain is equal to one whenever the relative phase between all of the elements is equal to zero – this is the case for an ideal array. Thus, the nominal array gain is equal to one. We now define a new random variable, \( G_{\text{loss}} \), as the array gain subtracted from one. \( G_{\text{loss}} \) represents the gain loss due to random phase shifts \( \Psi = [\Psi_1, ..., \Psi_N]^T \), namely

\[ G_{\text{loss}} \approx \frac{1}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\Psi_i - \Psi_j)^2. \]  

If we now let

\[ \Psi' = \begin{bmatrix} \Psi_1 - \Psi_2 \\ \Psi_1 - \Psi_3 \\ \vdots \\ \Psi_1 - \Psi_N \\ \Psi_2 - \Psi_3 \\ \vdots \\ \Psi_2 - \Psi_N \\ \vdots \\ \Psi_{N-1} - \Psi_N \end{bmatrix}, \quad (3.9) \]

where \( A \) is an \( M \times N \) matrix, then we can express \( G_{\text{loss}} \) in a more compact form as

\[ G_{\text{loss}} \approx \frac{\Psi'^T \Psi'}{N^2} = \frac{\Psi'^T A^T A \Psi}{N^2} = \frac{\Psi'^T B \Psi}{N}, \]  

where \( B \) is a matrix that depends on the specific form of the array and the phase shifts.
where $\mathbf{B} = \mathbf{A}^T \mathbf{A} / N$ is an $N \times N$ matrix. It can be shown that $\mathbf{B}$ is a symmetric idempotent matrix with rank $N - 1$. Therefore, through eigen-decomposition $\mathbf{B}$ can be written as $\mathbf{B} = \mathbf{P}^T \mathbf{A} \mathbf{P}$ where $\mathbf{P}$ is an $N \times N$ orthogonal matrix and $\mathbf{A}$ is an $N \times N$ diagonal matrix with one eigenvalue equal to zero and $N - 1$ eigenvalues equal to one [11]. The expression for the gain loss greatly simplifies to

$$G_{\text{loss}} \approx \frac{\Psi^T \mathbf{B} \Psi}{N} = \frac{\Psi^T \mathbf{P}^T \mathbf{A} \mathbf{P} \Psi}{N} = \frac{1}{N} \sum_{i=1}^{N-1} \Psi_i^2. \quad (3.11)$$

From (3.8) and (3.11) we see that the gain loss distribution is estimated as the sum of squares of random variables. As expected, the distribution of the gain loss depends on the distributions of the phases.

### 3.2 Estimation of the Distribution of the Gain Loss

We assume that the random phases $\Psi_1, ..., \Psi_N$ are independent and each normally distributed about zero with variance $\sigma_{\psi}^2$. With this assumption, we derive two estimates for the distribution of the gain loss. The first estimate of the gain loss distribution that we derive is a gamma distribution. This distribution is valid for arrays with any number of elements. The second estimate of the gain loss distribution is derived by following the literature (e.g. [2], [17], [8]), using the central limit theorem under the assumption of a large number of elements.

#### 3.2.1 Gamma Distribution Estimate of the Gain Loss

We begin by standardizing the random variables $\Psi_1, ..., \Psi_N$ as

$$Z_n = \frac{\Psi_n}{\sigma_{\psi}}. \quad (3.12)$$

From equation (3.11), the gain loss is written as the sum of $N - 1$ squared standard normal random variables scaled by $\sigma_{\psi}^2 / N$, namely

$$G_{\text{loss}} \approx \frac{1}{N} \sum_{i=1}^{N-1} (\sigma_{\psi} Z_n)_i^2 = \frac{\sigma_{\psi}^2}{N} \sum_{i=1}^{N-1} Z_n^2. \quad (3.13)$$

The sum of squares of $k$ independent standard normal random variables is a Chi-squared distribution with $k$ degrees of freedom [3]. Furthermore, a Chi-squared distribution scaled by some constant $c$ is a gamma distribution with shape $k/2$ and scale $2c$ [15]. Therefore, an approximate
distribution for the gain loss is given by a gamma distribution with shape \((N - 1)/2\) and scale \(2\sigma^2_\psi/N\).

### 3.2.2 Scaled Non-central Chi-squared Estimate of the Gain Loss

The distribution of the gain loss can be derived under the assumption that there are many elements in the array, allowing the use of the central limit theorem. It was shown in [2] that the distribution of the amplitude of the far-field radiation pattern \(V = |F_A(\theta, \psi)|\) is approximately given by a normal distribution with mean and variance given respectively by

\[
\mu_v = C_\psi(1) \sum_N |a_n|, \text{ and } \tag{3.14}
\]

\[
\sigma^2_v = \frac{1}{2} \left( 1 + C_\psi(2) - 2(C_\psi(1))^2 \right) \sum_N |a_n|^2. \tag{3.15}
\]

Here \(C_\psi(t)\) is the characteristic function of the random variable \(\Psi\), and \(a_n\) is the magnitude of the feed coefficient of element \(n\). Assuming, as we have done throughout this chapter, that the phase is normally distributed with zero mean and variance \(\sigma^2_\psi\), the characteristic function of \(\Psi\) is given by

\[
C_\psi(t) = e^{-\frac{1}{2} \sigma^2_\psi t^2}. \tag{3.16}
\]

Assuming further that the magnitude of the relative feed coefficient for each element is equal to one, the mean and variance of the amplitude of the far-field radiation pattern are calculated as

\[
\mu_v = e^{-\frac{1}{2} \sigma^2_\psi} \sum_N 1 = Ne^{-\frac{1}{2} \sigma^2_\psi}, \text{ and } \tag{3.17}
\]

\[
\sigma^2_v = \frac{1}{2} \left( 1 + e^{-\frac{1}{2} \sigma^2_\psi 2^2} - 2 \left( e^{-\frac{1}{2} \sigma^2_\psi} \right)^2 \right) \sum_N 1^2 = \frac{N}{2} \left( 1 + e^{-2\sigma^2_\psi} - 2 e^{-\sigma^2_\psi} \right). \tag{3.18}
\]

From equations (3.17) and (3.18), as the variance of the phase \(\sigma^2_\psi\) goes to zero, the mean of the far-field amplitude distribution goes to \(N\) and its variance goes to zero; this limit behavior is precisely what we expected.

Next we define a new random variable \(W\),

\[
W = \frac{V^2}{\sigma^2_\psi}. \tag{3.19}
\]
Since $V$ is normally distributed, $W$ is distributed as a non-central Chi-squared distribution \([5]\) with one degree of freedom and non-centrality parameter, $\gamma$, given by

$$
\gamma = \frac{\mu^2}{\sigma^2} = \frac{N^2 e^{-\sigma^2 \psi} - 2 e^{-\sigma^2 \psi}}{1 + e^{-2 \sigma^2 \psi} - 2 e^{-\sigma^2 \psi}} \approx 2 N e^{-\sigma^2 \psi}.
$$

(3.20)

We have defined $W$ in (3.19) because the non-central Chi-squared distribution has a well known cumulative distribution function as well as a well defined mean and variance. The cumulative distribution function of $W$ is given by the Marcum Q-function of order $1/2$ \([12]\). The mean and variance of $W$ are respectively $1 + \gamma$ and $2(1 + 2 \gamma)$ where $\gamma$ is the non-centrality parameter defined above \([1]\). The gain loss distribution is given by the resulting distribution from scaling $W$ by $\sigma^2 / N^2$ and subtracting it from one,

$$
G_{Loss} = 1 - G_A = 1 - \frac{V^2}{N^2} = 1 - \frac{\sigma^2}{N^2} W.
$$

(3.21)

### 3.3 Moments of the Gain Loss

#### 3.3.1 Gamma Distribution Estimate of the Gain Loss

The distribution of the gain loss given in equation (3.13) allows us to calculate its moments. Here we show calculations of both the mean and variance of the gain loss using the distribution derived in section 3.2.1,

$$
\mathbb{E}(G_{loss}) = \mathbb{E} \left( \frac{\sigma^2}{N} \sum_{i=1}^{N-1} Z_i^2 \right) = \frac{\sigma^2}{N} \sum_{i=1}^{N-1} \mathbb{E}(Z_i^2) = \frac{N - 1}{N} \sigma^2. \quad (3.22)
$$

In (3.22) we have used the linearity property of expectation and the fact that $\mathbb{E}(Z^2) = 1$ for a standard normal random variable $Z$. Similarly, we calculate the variance of the gain loss as

$$
\text{Var}(G_{loss}) = \text{Var} \left( \frac{\sigma^2}{N} \sum_{i=1}^{N-1} Z_i^2 \right) = \frac{\sigma^4}{N^2} \sum_{i=1}^{N-1} \text{Var}(Z_i^2) = \frac{N - 1}{N^2} 2 \sigma^4. \quad (3.23)
$$

In (3.23) we have used the fact that random variables $Z_1, \ldots, Z_N$ are independent and therefore the variance of their sum is equal to the sum of their variances; that for a random variable $X$ and scalar $a$, $\text{Var}(aX) = a^2 \text{Var}(X)$; and that for a normal random variable $Z$, $\text{Var}(Z^2) = \mathbb{E}(Z^4) - (\mathbb{E}(Z^2))^2 = 3 - 1 = 2$.

More generally, we can compute any moment of the distribution given by (3.8) by first computing its moment generating function. As we have stated in 3.2.1, the distribution of the gain loss is a
gamma distribution and hence its moment-generating function is \[15\]

\[M_{G_{\text{loss}}}(t) = E\left[e^{tG_{\text{loss}}}\right] = \left(1 - \frac{2\sigma_v^2}{N} t\right)^{\frac{N-1}{2}} \quad \forall \quad t < \frac{N}{2\sigma_v^2}. \tag{3.24}\]

### 3.3.2 Scaled Non-central Chi-squared Estimate of the Gain Loss

We calculate the mean and variance of the gain loss assuming that it takes the distribution derived in section 3.2.2. First, the expected value of the gain loss is given by

\[E[G_{\text{loss}}] = E[1 - G_A] = 1 - E[G_A] = 1 - E\left[\frac{|F_A|^2}{N^2}\right] = 1 - E\left[\frac{V^2}{N^2}\right] = 1 - \frac{\sigma_v^2}{N^2} E[W] \tag{3.25}\]

\[= 1 - \frac{\sigma_v^2}{N^2} (1 + \gamma) = 1 - \frac{\sigma_v^2}{N^2} - \frac{\mu_v^2}{N^2},\]

where \(\mu_v^2\) and \(\sigma_v^2\) are given by equations (3.17) and (3.18). Next, we calculate the variance of \(G_{\text{loss}}\), namely

\[\text{Var}(G_{\text{loss}}) = \text{Var}(1 - G_A) = \text{Var}(G_A) = \text{Var}\left(\frac{V^2}{N^2}\right) = \frac{\sigma_v^4}{N^4} \text{Var}\left(\frac{V^2}{\sigma_v^2}\right) \tag{3.26}\]

\[= \frac{\sigma_v^4}{N^4} \text{Var}(W) = \frac{\sigma_v^4}{N^4} 2(1 + 2\gamma) = \frac{2\sigma_v^4}{N^4} + \frac{4\sigma_v^2\mu_v^2}{N^4}.\]

The moments of a distribution, particularly the first two, are quite useful for comparing and characterizing distributions. In section 5.1, we will use equations (3.22), (3.25), (3.23), and (3.26) to compare the two gain loss distribution estimates that we have derived with one another as well as with simulations.
4. Application of Gain Expressions to the Design of Phased Array Systems

Beyond estimating the moments of the gain loss distribution, we are also interested in exploring how knowledge of the gain loss distribution may be applied to the design of a phased array system. Of particular interest are the restrictions on the variance of the random element spacing needed if we wish to limit the gain loss of the main lobe. In this chapter, we first derive an expression for the variance of the phase of each element in an array as a function of the gain loss and the probability that this gain loss is realized. We then show how error in the relative spacing of the elements will result in error in their relative phases. Finally, by relating the variance of the element position to the variance of the phase, we derive an expression for the element position variance that is a function of the gain loss and the probability that this gain loss is realized.

4.1 Phase Variance

We again consider phase angles $\Psi_1, ..., \Psi_N$ that are normally distributed with zero mean and standard deviation $\sigma_\psi$. Let $p$ be the probability that the gain loss is less than or equal some value $g$; both $p$ and $g$ are between 0 and 1.

\[ \Psi_n \sim \mathcal{N}(0, \sigma_\psi^2) \forall n \in [1, N], \text{ and} \]
\[ p = \mathbb{P}(G_{\text{loss}} \leq g). \]  

By substituting equation (3.13) into (4.2), we estimate $p$ as

\[ p \cong \mathbb{P} \left( \frac{\sigma^2}{N} \sum_{i=1}^{N-1} Z_i^2 \leq g \right) = \mathbb{P} \left( \sum_{i=1}^{N-1} Z_i^2 \leq \frac{gN}{\sigma_\psi^2} \right) = F_{\chi^2} \left( \frac{gN}{\sigma_\psi^2}; N - 1 \right). \]  

Here $F_{\chi^2}$ is the Chi-squared cumulative distribution function (CDF). Taking the inverse Chi-squared CDF and solving for $\sigma_\psi^2$ results in a convenient expression for the variance of $\Psi_1, ..., \Psi_N$ needed to limit the gain loss to $g$ with probability $p$. The value of the variance is given by

\[ \sigma_\psi^2 \cong \frac{gN}{F_{\chi^2}^{-1}(p; N - 1)}. \]
Since the gain loss will increase as $\sigma_\psi$, we can state that as long as $\sigma_\psi$ is less than the value given by (4.4), the gain loss will be no more than $g$ with probability $p$. Importantly, in order to use the expression in (4.4) for a given array, we need to show that $\Psi_1, ..., \Psi_N$ are (a) normally distributed, (b) have zero mean, (c) each have the same variance, and (d) said variance is reasonably small.

4.2 Expression for the Position Variance

We apply expression (4.4) to common array architectures where the array elements are positioned with normally distributed error in either one, two or all three directions. We start by examining how (4.4) can be applied in general to an array with arbitrarily spaced elements whose positions have normally distributed error. The analysis will proceed as follows: first we use (2.24) to calculate the relative phase between the elements, then we show that under certain conditions the relative phases are normally distributed with zero mean and the same variances, and finally we use (4.4) to find the necessary limits that must be placed on the position variance to limit the gain loss to a specified amount. We then examine how this expression simplifies for array architectures that are generally steered only in one direction such as the uniform linear array.

Suppose we have an $N$ element array such that each element $n \in [1, ..., N]$ has a relative displacement of $d_n + \vec{D}_n$ where $d_n$ is the nominal displacement of the element and $\vec{D}_n$ represents the error in the element’s position. Assume also that each $\vec{D}_n$ is a 3-element vector of independent normal random variables $X_n$, $Y_n$ and $Z_n$ each with zero mean and variances of $\sigma_x^2$, $\sigma_y^2$ and $\sigma_z^2$ respectively. Assuming that the array is steered with cophasal element excitation, we calculate from equation (2.24) the relative phase between the elements as

$$
\Psi_n(\hat{r}) = k(\hat{r} \cdot (d_n + \vec{D}_n)) - k(\hat{r}_0 \cdot d_n) = k(\hat{r} \cdot d_n) - k(\hat{r}_0 \cdot d_n) + k(\hat{r} \cdot \vec{D}_n). \quad (4.5)
$$

It can be shown that for some normal random variable $Q$ with mean $\mu_q$ and variance $\sigma_q^2$, if $a$ and $b$ are scalars, then the random variable $S = aQ + b$ is normally distributed with mean $a\mu_q + b$ and variance $a^2\sigma_q^2$ [3]. Further, for some normally distributed random variable $R$ with mean $\mu_r$ and variance $\sigma_r^2$, if $S = Q + R$, then $S$ is normally distributed with mean $\mu_q + \mu_r$ and variance $\sigma_q^2 + \sigma_r^2$ [3]. These properties of normal random variables show that $\Psi_n(\hat{r})$ in equation (4.5) is normally distributed. Since the mean of each element in $\vec{D}_n$ is assumed to be zero, the mean of $\Psi_n(\hat{r})$ is
In order to calculate the variance of $\Psi_n(\hat{r})$, we need to expand $k(\hat{r} \cdot \vec{D}_n)$ as

$$k(\hat{r} \cdot \vec{D}_n) = k(X_n \sin(\theta) \sin(\phi) + Y_n \sin(\theta) \cos(\phi) + Z_n \cos(\theta)). \quad (4.6)$$

Hence the variance of $\Psi_n(\hat{r})$ is equal to

$$\sigma^2_{\psi} = k^2(\sigma_x^2 \sin^2(\theta) \sin^2(\phi) + \sigma_y^2 \sin^2(\theta) \cos^2(\phi) + \sigma_z^2 \cos^2(\theta)) = k^2 \hat{r}^T \Sigma_d \hat{r}. \quad (4.7)$$

Here $\Sigma_d$ is the covariance matrix of the multivariate normal random variable $\vec{D}_n$. Since the elements of $\vec{D}_n$ are assumed to be independent, its covariance matrix is a diagonal matrix with $\sigma_x^2$, $\sigma_y^2$, and $\sigma_z^2$ along the diagonal. To summarize, we have that

$$\Psi_n(\hat{r}) \sim N(k(\hat{r} \cdot \vec{d}_n) - k(\hat{r}_0 \cdot \vec{d}_n), k^2 \hat{r}^T \Sigma_d \hat{r}). \quad (4.8)$$

This is not quite the form we need in order to apply (4.4). While we have shown that each $\Psi_n$ is normally distributed with the same variance, we have also shown that the mean of $\Psi_n$ is only equal to zero whenever $\hat{r} \cdot \vec{d}_n = \hat{r}_0 \cdot \vec{d}_n$. Nonetheless, we apply equation (4.4) when speaking of the gain of the main beam since the main beam will ideally be in the steered direction (i.e., $\hat{r} = \hat{r}_0$.) Thus, in the steered direction we have that

$$\Psi_n(\hat{r}_0) \sim N(0, k^2 \hat{r}_0^T \Sigma_d \hat{r}_0). \quad (4.9)$$

Next we substitute the variance of $\Psi_n(\hat{r}_0)$ into (4.4) and simplify as

$$\hat{r}_0^T \Sigma_d \hat{r}_0 \approx \frac{gN}{k^2 F^{-1}_{\chi^2}(p; N-1)}. \quad (4.10)$$

Expression (4.10) can be further simplified for certain array geometries, such as the uniform linear array, which is generally only steered along its azimuth. For the uniform linear array shown in figure 2.2, $\phi_0$ will be fixed at $\pi/2$. Hence the variance, as calculated in equation (4.7), simplifies in this case to

$$\sigma^2_x \sin^2(\theta_0) + \sigma^2_z \cos^2(\theta_0) \approx \frac{gN}{k^2 F^{-1}_{\chi^2}(p; N-1)}. \quad (4.11)$$

Equation (4.10) can also be simplified if the variance of the error in each direction is the same.
If $\sigma_x^2 = \sigma_y^2 = \sigma_z^2$ in equation (4.7), then the variance of $\Psi_n(\hat{r})$ is $k^2\sigma_x^2$. For this case we have

$$\sigma_x^2 \cong \frac{gN}{k^2 F^{-1}_{\chi^2}(p, N - 1)}.$$  \hspace{1cm} (4.12)

This assumption makes simulation and evaluation of the position variance expression simpler since it eliminates a number of variables.
5. Simulations of Results

5.1 Gain Loss Distribution

In this chapter we show simulations performed in order to assess the accuracy of the gamma distribution estimate of the gain loss given by (3.13). In addition, we compare it to the scaled non-central Chi-squared estimate of the gain loss given by (3.21). We also evaluate our expression for the position variance given by equation (4.10).

We used Monte Carlo simulations of a uniform linear array system to generate an estimate of the distribution of the gain loss. We then graphed the probability density function from the distribution estimate of the gain loss given by (3.13) against the simulated distribution. The results are shown in figures 5.1, 5.2, and 5.3.

Figure 5.1 shows the distribution estimate of the gain loss given by (3.13) (blue) versus the simulated distribution (black) for three different values of $N$, where $N$ is the number of elements in the array. The distribution estimate of the gain loss given by (3.13) is quite close to the simulated distribution. The distribution estimate of the gain loss given by (3.13) actually seems to fit slightly better for smaller arrays than for larger arrays.

Figure 5.2 shows the distribution estimate of the gain loss given by (3.13) versus the simulated distribution, this time for three different values of $\theta_0$, where $\theta_0$ is the steered direction of the array. In each of these cases, and for other values of $\theta_0$ as well, the distribution estimate of the gain loss given by (3.13) fits the simulated distribution almost exactly. Steering the array in different directions does not seem to affect the accuracy of the distribution estimate of the gain loss given by (3.13). However, from figure 5.3 we do see that as the element position deviation grows large, the distribution estimate of the gain loss given by (3.13) does become quite inaccurate. This is most likely due to the fact that we have used the small angle approximation in our derivation. When the position deviation is relatively large, the resulting phase error is no longer close to zero and the small angle approximation is no longer valid.

We have also compared the distribution estimate of the gain loss given by (3.13) to that given by (3.21) in several ways. First, we show in figure 5.4 that for an array with only a few elements, the distribution estimate of the gain loss given by (3.13) is almost exactly the same as the simulated.\footnote{It is somewhat difficult to see the simulated distribution since it is directly behind the distribution estimate of the gain loss given by (3.13) shown in blue.}
Figure 5.1: Distribution of the gain loss of a ULA whose elements have position error along the axis of the array for three different values of $N$. The distribution estimate of the gain loss given by (3.13) is shown in blue, the simulated distribution is shown in black.

Figure 5.2: Distribution of the gain loss of a ULA whose elements have position error along the axis of the array for three different values of $\theta_0$. The distribution estimate of the gain loss given by (3.13) is shown in blue, the simulated distribution is shown in black.
Figure 5.3: Distribution of the gain loss of a ULA whose elements have position error along the axis of the array for three different values of $\sigma_x$. The distribution estimate of the gain loss given by (3.13) is shown in blue, the simulated distribution is shown in black.

As the number of elements in the array grows larger, the distribution estimate of the gain loss given by (3.21) begins to converge onto the simulated distribution.

As another way of comparing the distribution estimate of the gain loss given by (3.13) and (3.21), we have calculated the mean and variance of each distribution for a uniform linear array with normally distributed element spacing errors. We then performed a Monte Carlo simulation of the system and estimated the mean and variance. We found that for arrays with only a few elements, the mean and variance of the distribution estimate of the gain loss given by (3.13) agree very closely with those of the simulated distribution. However, the mean and variance of the distribution estimate of the gain loss given by (3.21) were not nearly as accurate when $N$ was small. These results are shown in figures 5.5 and 5.6.

5.2 Evaluation of the Position Variance Expression

We conclude this chapter by evaluating the accuracy of the position variance expression of equation (4.10) that we derived in chapter 4. Our method for analyzing the position variance expression is as follows. We pick $p$ and $g$ as if we were trying to design an array whose gain loss due to element position errors is no more than $g$ with probability $p$. Here we assume that we have a uniform linear
Figure 5.4: A comparison of the two estimated distributions of the gain loss derived in section 3.2 for different array sizes. The distribution estimate of the gain loss given by (3.13) is shown in blue, the distribution estimate of the gain loss given by (3.21) is shown in red, and the simulated distribution is shown in black. These distributions are shown for a ULA with $N = 3$ elements (top), $N = 5$ elements (middle), and $N = 25$ elements (bottom). The first estimate is much better for small array sizes; for large arrays, all three distributions are nearly identical.
Figure 5.5: The mean gain loss as a function of the number of elements in the array. The blue line shows the mean gain loss calculated by averaging over a large number of samples ($\sim 10^6$). The green line shows the mean gain loss calculated from the gamma distribution estimate of the gain loss; given by equation (3.22). The red line shows the mean gain loss calculated from the scaled non-central Chi-squared distribution estimate of the gain loss; given by equation (3.25).
Figure 5.6: The standard deviation of the gain loss as a function of the number of elements in the array. The blue line shows the standard deviation of the gain loss calculated by averaging over a large number of samples (~ $10^6$). The green line shows the standard deviation of the gain loss calculated from the gamma distribution estimate of the gain loss; given by equation (3.22). The red line shows the standard deviation of the gain loss calculated from the scaled non-central Chi-squared distribution estimate of the gain loss; given by equation (3.25).
array whose elements have error in all three directions with equal variances. Then from equation (4.11), we calculate the allowed variance for this set of $p$ and $g$.

We then generate normal random variables with this variance and simulate the system in order to estimate the resulting gain loss. We did so for a large number of trials ($\sim 10^6$) and tabulated the resulting gain losses. Finally, we found the first value of gain loss that is greater than $100 \times p$ percent of the other gain loss values. If we denote the number of trials by $T$, then this amounts to sorting the tabulated gain losses in ascending order and selecting the $[Tp]$-th element. Ideally, that is if our expression is accurate, this value of gain loss will be equal to $g$.

We are interested in seeing how well our position variance expression (4.10) performs, particularly as $N$ varies. We have fixed $p$ to be 95% and $g$ to be 10% and carried out the above process for a range of values for $N$. The results are shown in figure 5.7. The resulting maximum gain loss only deviates about one-tenth of a percent from the designed maximum gain loss. Thus, we were successfully able to limit the gain loss of the array with a fairly high level of confidence by restricting the variance of the element spacing errors. Furthermore, the deviation of the actual maximum gain loss from the designed maximum gain loss does not substantially change as the number of elements in the array varies. This observation tells us that our position variance expression appears to be accurate regardless of the size of the array.
Figure 5.7: A plot of the resultant maximum gain loss (blue) and the designed maximum gain loss (black dotted line) by use of the position variance expression given by equation (4.10) as a function of the number of elements in the array.
6. Conclusions

We have shown that when the relative spacing between the antenna elements is normally distributed, the resulting relative electrical phase between the elements is also normally distributed. Using this result, we were able to derive two different estimates for the distribution of the gain loss of an array with randomly spaced elements.

The first distribution we derived (3.13) is valid for arrays with any number of elements – this distribution is a gamma distribution. We derived a second distribution (3.21) using the central limit theorem under the assumption of a large number of elements in the array. As we expected, simulations show that the first distribution is much closer to the simulated gain loss distribution for arrays with small numbers of elements.

We also found an expression (4.10) for the allowed variance in the element position as a function of gain loss, the number of elements of the array, and the probability that this gain loss is realized. Our simulations show that our expression for the position variance is accurate, even for arrays with small numbers of elements. Using this expression a designer can specify placement tolerance as a function of the quality (in terms of gain in the desired direction) of the array. Our results are limited to the assumption that the element position errors are normally distributed with zero mean and a small variance.
Bibliography


