CONSERVATIVE DILATIONS OF DISSIPATIVE MULTIDIMENSIONAL SYSTEMS: THE COMMUTATIVE AND NON-COMMUTATIVE SETTINGS

JOSEPH A. BALL AND DMITRY S. KALIUZHNYI-VERBOVETSKYI

ABSTRACT. We establish the existence of conservative dilations for various types of dissipative non-commutative N-dimensional (N-D) systems. As a corollary, a criterion of existence of conservative dilations for corresponding dissipative commutative N-D systems is obtained. We point out the cases where this criterion is always fulfilled, and the cases where it is not always fulfilled.

1. Introduction

Consider *1D linear system* of the form

$$\Sigma : \left\{ \begin{array}{rcl} x(z) & = & zAx(z) + zBu(z), \\ y(z) & = & Cx(z) + Du(z), \end{array} \right. \tag{1.1}$$

where $x \in \mathcal{X}[[z]],\ u \in \mathcal{U}[[z]],\ y \in \mathcal{Y}[[z]]$ are formal power series in a single indeterminate z with the coefficients from separable Hilbert spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y}$, respectively, and $A \in \mathcal{L}(\mathcal{X}),\ B \in \mathcal{L}(\mathcal{U}, \mathcal{X}),\ C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}),\ D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, i.e., A, B, C, and D are bounded linear operators acting in the corresponding pairs of spaces. Comparing the coefficients of formal power series in the two sides of the equalities (1.1), we obtain the standard form of *1D linear system in a discrete time domain*:

$$\Sigma : \left\{ \begin{array}{rcl} x_{j+1} & = & Ax_j + Bu_j, \\ y_j & = & Cx_j + Du_j, \end{array} \right. \qquad j = 0, 1, \dots, \tag{1.2}$$

together with the zero initial condition $x_0 = 0$. Here and in the sequel we prefer to write a linear system in the form of equations for formal power series in one or several, commuting or non-commuting, indeterminates.

From (1.1) we obtain the equality $y(z) = T_\Sigma(z)u(z)$, where

$$T_\Sigma(z) := D + \sum_{j=1}^{\infty} CA^{j-1}Bz^j \in \mathcal{L}(\mathcal{U}, \mathcal{Y})[[z]]$$

is a formal power series in z with the coefficients from $\mathcal{L}(\mathcal{U}, \mathcal{Y})$, which is called the *transfer function* of the system Σ. It is clear that substitution of scalars for the indeterminate in the series T_Σ is well defined in some neighborhood of $0 \in \mathbb{C}$ where the series converges, so that T_Σ becomes a holomorphic function on this neighborhood where

$$T_\Sigma(z) = D + C(I_\mathcal{X} - zA)^{-1}zB.$$

If a system Σ of the form (1.1) is *dissipative*, i.e., the system operator

$$U := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}) \tag{1.3}$$

is contractive, or, equivalently, system trajectories $\{u_j, x_j, y_j\}_{j=0,1,\dots}$ (i.e., solutions of the system equations (1.2)) satisfy the energy-balance inequality

$$\|x_{j+1}\|^2 - \|x_j\|^2 \le \|u_j\|^2 - \|y_j\|^2 \text{ for } j = 0, 1, \dots \text{ (instantaneous form)}$$
$$\|x_{N+1}\|^2 - \|x_0\|^2 \le \sum_{j=0}^{N} \|u_j\|^2 - \sum_{j=0}^{N} \|y_j\|^2 \text{ for } N = 0, 1, \dots \text{ (integral form)}, \quad (1.4)$$

then T_Σ belongs to the *Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ of holomorphic contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on the unit disk $\mathbb{D}$. Conversely, every formal power series F from the class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ has a dissipative, moreover conservative realization, i.e., a 1D system Σ with a unitary operator U from (1.3), or equivalently, a 1D system Σ so that the system trajectories $\{(u_j, x_j, y_j)\}_{j=0,1,\dots}$ satisfy the energy-balance equality

$$\|x_{j+1}\|^2 - \|x_j\|^2 = \|u_j\|^2 - \|y_j\|^2 \text{ for } j = 0, 1, \dots \text{ (instantaneous form)}$$
$$\|x_{N+1}\|^2 - \|x_0\|^2 = \sum_{j=0}^{N} \|u_j\|^2 - \sum_{j=0}^{N} \|y_j\|^2 \text{ for } N = 0, 1, \dots \text{ (integral form)} \quad (1.5)$$

with a similar identity for trajectories of the adjoint system, such that $F = T_\Sigma$.

If one replaces the second equation in (1.1) by

$$y(z) = zCx(z) + zDu(z),$$

then one obtains a 1D system Σ' which corresponds to a *1D system with a unit delay* in a discrete time domain:

$$\Sigma' : \begin{cases} x_j &= Ax_{j-1} + Bu_{j-1}, \\ y_j &= Cx_{j-1} + Du_{j-1}, \end{cases} \qquad j = 1, \dots, \quad (1.6)$$

and has the transfer function

$$T_{\Sigma'}(z) = zD + zC(I_{\mathcal{X}} - zA)^{-1}zB = zD + \sum_{j=2}^{\infty} CA^{j-2}Bz^j = zT_\Sigma(z).$$

In the 1D case this gives an equivalent system theory, e.g., the transfer function of a dissipative system Σ' (which corresponds to the same contractive system operator (1.3) as does Σ) belongs to the subclass $\mathcal{S}^0(\mathcal{U}, \mathcal{Y})$ of the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ consisting of functions vanishing at $0 \in \mathbb{C}$, and by the Schwarz lemma $\mathcal{S}^0(\mathcal{U}, \mathcal{Y}) = z\mathcal{S}(\mathcal{U}, \mathcal{Y})$. However, in the N-D case counterparts of systems Σ and Σ' lead to non-equivalent theories: e.g., there is no canonical isomorphism between the (commutative) *Schur–Agler class* $\mathcal{SA}_N(\mathcal{U}, \mathcal{Y})$ and its subclass $\mathcal{SA}_N^0(\mathcal{U}, \mathcal{Y})$ consisting of functions vanishing at $0 \in \mathbb{C}^N$, which serve as the corresponding classes of transfer functions of conservative N-D systems (see [1, 12, 11] and [19]).

Denote the collection of data for the 1D system (1.1) or (1.2) (which is the same for system (1.6)) by $\Sigma = (1; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, or in more detail, by

$$\Sigma = (1; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}).$$

We shall identify the system Σ (say, of the form (1.1)) with this collection of data. Recall that the 1D system $\widetilde{\Sigma} = (1; \widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is called a *dilation* of the 1D system $\Sigma = (1; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ (or, alternatively, we say that Σ is a *reduction* of $\widetilde{\Sigma}$) if there exist subspaces $\mathcal{D}$ and $\mathcal{D}_*$ in $\widetilde{\mathcal{X}}$ such that

$$\widetilde{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_*, \quad (1.7)$$

$$\widetilde{A}\mathcal{D} \subset \mathcal{D}, \quad \widetilde{C}\mathcal{D} = \{0\}, \quad \widetilde{A}^*\mathcal{D}_* \subset \mathcal{D}_*, \quad \widetilde{B}^*\mathcal{D}_* = \{0\}, \quad (1.8)$$

$$A = P_{\mathcal{X}}\widetilde{A}|\mathcal{X}, \quad B = P_{\mathcal{X}}\widetilde{B}, \quad C = \widetilde{C}|\mathcal{X} \tag{1.9}$$

(here $P_{\mathcal{X}}$ denotes the orthogonal projection onto $\mathcal{X}$ in $\widetilde{\mathcal{X}}$). It is easily seen that in this case $\widetilde{\Sigma}$ and Σ have the same transfer functions ($T_{\widetilde{\Sigma}}(z) = T_{\Sigma}(z)$). In the affine setting (where no energy-balance relations as in (1.4) are taken into account), the process of reduction arises naturally in the Kalman theory of constructing a system Σ with minimal possible state-space dimension having a given rational matrix function F as its transfer function from a given system Σ having transfer function equal to F (see [18]). Less well known is that there is also a version of Kalman reduction which takes into account the energy-balance relations (1.4) (see [6, 7]). The key notion is that of optimal and of $*$-optimal dissipative linear system: among all dissipative realizations of a given Schur-class function $S(z)$, the optimal (respectively, $*$-optimal) dissipative realization is such that the state to which a given finite input-string drives the system has minimum possible (respectively maximum possible) norm (or energy)—to avoid trivialities one considers only observable realizations for the $*$-optimal case. Given a conservative realization $\widetilde{\Sigma}$, one can always reduce to an optimal dissipative system Σ_o or a $*$-optimal dissipative system Σ_{o*}. In the theory of optimal and $*$-optimal dissipative systems, the converse direction of conservative dilation of a given (not necessarily optimal) dissipative system plays a key role.

The following lemma which has been proved in [20, Lemma 2.1] gives a reformulation of this geometrical definition (1.8) of dilation/reduction in algebraic language and generalizes the Sarason lemma [26, Lemma 0] on operator dilations to the case of system dilations.

Lemma 1.1. *The system* $\widetilde{\Sigma} = (1; \widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *is a dilation of the system* $\Sigma = (1; A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}} \supset \mathcal{X}$ *and for all* $j \in \mathbb{Z}_+$ *the following equalities hold:*

$$A^j = P_{\mathcal{X}}\widetilde{A}^j|X, \ A^jB = P_{\mathcal{X}}\widetilde{A}^j\widetilde{B}, \ CA^j = \widetilde{C}\widetilde{A}^j|\mathcal{X}, \ CA^jB = \widetilde{C}\widetilde{A}^j\widetilde{B}. \tag{1.10}$$

A special case of Lemma 1.1 where $\mathcal{U} = \mathcal{Y} = \{0\}$ is the Sarason lemma [26, Lemma 0]. From Lemma 1.1 we obtain the well known fact (see, e.g., [5]) that transfer functions of a 1D system and of its dilation coincide (as formal power series, and as functions holomorphic at $0 \in \mathbb{C}$, on some neighborhood of 0). It has been proved in [5] that any dissipative 1D system has a conservative dilation; moreover, there is a minimal such conservative dilation which is unique up to a unitary similarity. This result is a generalization of the Sz.-Nagy theorem on the existence of a unitary dilation of a contractive operator [28].

The purpose of the present paper is to extend all these ideas, especially the conservative dilation theorem for dissipative systems from [5, 6, 7], to the case of N-D systems, both in the commutative and in the non-commutative settings. As was mentioned above, the two types of systems (1.2) and (1.6), while equivalent in the 1-D case, diverge in several directions in the N-D case. In Section 2 we introduce the various types of N-D systems (for both the commutative and non-commutative case) which we shall consider here, namely: Kalyuzhniy-Verbovetskyi (KV) systems (following the terminology from [11] for the systems introduced in [19]), Fornasini-Marchesini (FM) systems (appearing in [15] for the commutative case), Givone-Roesser (GR) systems (originating in [16, 17] for the commutative case) and structured or Ball-Groenewald-Malakorn (BGM) systems (appearing in

[9, 10] for the noncommutative case). Here we also delineate the various relations among these families of systems and introduce their respective transfer functions.

In Section 3 we define the notion of dissipative/conservative for each of these types of N-D systems by demanding that each 1-D system in an appropriate parametrized family of 1-D systems associated with the N-D system be dissipative/conservative as a 1-D system in the sense described above; this follows the path of [20, 21] rather than that of [13, 14, 10] where the dissipative/conservative property was understood via multidimensional analogues of the "time-domain" energy-balance relations (1.4) and (1.5).

As there is a multitude of types of N-D systems, there is a corresponding multitude of forms of the conservative system dilation theorems for the N-D case. The first theorem obtained of this kind was in the context of commutative KV systems (see [20]). It was shown in [20] that for $N \geq 3$ not every N-dimensional dissipative KV system has a conservative dilation, and a criterion for existence of a conservative dilation of dissipative KV system was obtained. Let us remark also that in [21] it was shown that any commutative KV system has a J-conservative dilation, with some signature operator $J = J^{-1} = J^*$ which defines a (in general, indefinite) metric in the state space of this dilation system. Section 4 presents our results on conservative dilations of KV systems. We prove that every dissipative non-commutative KV system has a (even so-called *uniform*) conservative dilation (see Theorem 4.5). As a consequence, we obtain the criterion from [20] for the existence of a conservative dilation for commutative KV systems (see Theorem 4.10). Moreover, in the case when a conservative dilation exists, we establish the existence of a so-called *uniform* conservative dilation for the commutative case (see Theorem 4.11); this is an improvement of the main result of [20].

In Section 5 we establish analogous results for Fornasini–Marchesini (FM) systems, both in the commutative and non-commutative settings (see Theorems 5.5, 5.11 and 5.12). Note that the version of dissipative FM systems considered here (which will be called the *polydisk version*), as well as of KV systems, is related to functions on the unit polydisk $\mathbb{D}^N$ in the commutative setting, and to functions on the *non-commutative polydisk* $\mathcal{D}^N$, the set of N-tuples of strictly contractive linear operators on a common separable Hilbert space, in the non-commutative setting. Conservative FM systems of this type were first considered in [11]. (The *ball version* of conservative FM systems was considered in [13] in the commutative setting, and in [14] in the non-commutative, or Cuntz-algebra, setting.)

In Section 6 we prove that every dissipative non-commutative structured N-D system, or Ball–Groenewald–Malakorn (BGM) system (see [9, 10]) has a conservative dilation (see Theorem 6.6). In the commutative setting, as opposed to the cases of KV and FM systems, we obtain that *every* dissipative BGM system has a conservative dilation (see Theorem 6.9). We remark that the notion of *commutative BGM system* appears here, apparently, for the first time.

One of the main tools used to obtain these conservative dilation theorems is a version of the Sarason lemma for each type of system which we also obtain along the way. In the derivation of many of the results mentioned above, the non-commutative case is derived first and then the commutative case is derived as an application of the non-commutative version.

As was mentioned above, the notion of conservative dilation of a dissipative system plays a key role in the theory of optimal and $*$-optimal dissipative systems.

We expect that the results of this paper will provide the starting point for the development of an N-D analogue of the theory of optimal and $*$-optimal dissipative linear systems.

2. Preliminaries on multidimensional systems

We consider N*-dimensional linear systems* of the form

$$\Sigma : \begin{bmatrix} x(z) \\ y(z) \end{bmatrix} = U_z \begin{bmatrix} x(z) \\ u(z) \end{bmatrix}, \tag{2.1}$$

where $x(z), u(z), y(z)$ are formal power series (FPSs) in N indeterminates $z = (z_1, \dots, z_N)$ which commute (in this case we write $\Sigma = \Sigma^{\rm c}$) or do not commute (in this case we write $\Sigma = \Sigma^{\rm nc}$), with the coefficients in separable Hilbert spaces $\mathcal{X}$ (the *state space*), $\mathcal{U}$ (the *input space*), $\mathcal{Y}$ (the *output space*), and

$$U_z = U_0 + \sum_{k=1}^{N} U_k z_k = \begin{bmatrix} A_z & B_z \\ C_z & D_z \end{bmatrix} \tag{2.2}$$

is a linear polynomial in z with the coefficients in $\mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$. We use the convention that the product of two FPSs with compatible spaces of (operator or vector) coefficients (in particular, one or both of these FPSs may be polynomial) is well defined, i.e., the indeterminates formally commute with the coefficients. In the commutative case,

$$x(z) = \sum_{t \in \mathbb{Z}_N^+} x_t z^t \in \mathcal{X}[[z_1, \dots, z_N]],$$

where $\mathcal{L}[[z_1, \dots, z_N]]$ denotes the linear space of commutative FPSs with the coefficients in the linear space $\mathcal{L}$, $\mathbb{Z}_N^+ := \{t \in \mathbb{Z}^N : t_k \geq 0, k = 1, \dots, N\}$ and $z^t := \prod_{k=1}^{N} z_k^{t_k}$ for $t = (t_1, \dots, t_N) \in \mathbb{Z}_N^+$. Similarly for $u(z) \in \mathcal{U}[[z_1, \dots, z_N]]$, $y(z) \in \mathcal{Y}[[z_1, \dots, z_N]]$.

In the non-commutative case,

$$x(z) = \sum_{w \in \mathcal{F}_N} x_w z^w \in \mathcal{X} \langle\langle z_1, \dots, z_N \rangle\rangle,$$

where $\mathcal{L}\langle\langle z_1, \dots, z_N \rangle\rangle$ denotes the linear space of non-commutative FPSs with the coefficients in the linear space $\mathcal{L}$, $\mathcal{F}_N$ is the free semigroup with N generators $g_1, \dots, g_N$ and neutral element $\emptyset$; $z^w := z_{i_1} \cdots z_{i_m}$ for $w = g_{i_1} \cdots g_{i_m}$, and $z^\emptyset = 1$. Similarly for $u(z) \in \mathcal{U} \langle\langle z_1, \dots, z_N \rangle\rangle$, $y(z) \in \mathcal{Y} \langle\langle z_1, \dots, z_N \rangle\rangle$.

The time-domain version of the system equations is obtained by equating power-series coefficients in the frequency-domain equations (2.1). For the noncommutative case the result is

$$x_{g_j v} = A_0 x_{g_j v} + A_j x_v + B_0 u_{g_j v} + B_j u_v, \tag{2.3}$$

$$y_{g_j v} = C_0 x_{g_j v} + C_j x_v + D_0 x_{g_j v} + D_j x_v \text{ for } j = 1, \dots, N \text{ and } v \in \mathcal{F}_N \tag{2.4}$$

while for the commutative case the result is

$$x_t = A_0 x_t + \sum_{j=1}^N A_j x_{t-e_j} + B_0 u_t + \sum_{j=1}^N B_j u_{t-e_j} \tag{2.5}$$

$$y_t = C_0 x_t + \sum_{j=1}^N C_j x_{t-e_j} + D_0 u_t + \sum_{j=1}^N D_j u_{t-e_j} \text{ for } t \in \mathbb{Z}_N^+ \tag{2.6}$$

where $e_j = (0, \dots, 1, \dots 0)$ is the j-th standard N-tuple for the lattice $\mathbb{Z}^N$. In the sequel we shall focus exclusively on the frequency-domain version of the system equations.

In this paper we concentrate on the following types of N-D systems.

2.1. **Kaliuzhnyi-Verbovetskyi (KV) systems.** These are systems (2.1) for which

$$U_z = U_z^{\mathrm{KV}} = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ z\mathbf{C} & z\mathbf{D} \end{bmatrix}, \tag{2.7}$$

where $z\mathbf{A} = \sum_{k=1}^N z_k A_k$ for $\mathbf{A} = (A_1, \dots, A_N) \in \mathcal{L}(\mathcal{X})^N$, and similarly for $z\mathbf{B}, z\mathbf{C}$ and $z\mathbf{D}$. The corresponding "N-D time-domain" equations are obtained as equations for the coefficients of FPSs in (2.1) with $U_z = U_z^{\mathrm{KV}}$ given in (2.7). In the commutative case, they were considered first in [19] (see also [20, 21, 22, 11]).

2.2. **Fornasini–Marchesini (FM) systems.** These are systems (2.1) for which

$$U_z = U_z^{\mathrm{FM}} = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ C & D \end{bmatrix}, \tag{2.8}$$

where C and D are constant. The corresponding "N-D time-domain" equations have appeared first in the commutative case in [15], and in the non-commutative case in [9].

2.3. **Givone–Roesser (GR) systems.** These are a special case of FM systems where

$$U_z = U_z^{\mathrm{GR}} = \begin{bmatrix} z\mathbf{P} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2.9}$$

with a constant operator-block matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}), \tag{2.10}$$

and $z\mathbf{P} := \sum_{k=1}^N z_k P_k$ (here $\mathbf{P} = (P_1, \dots, P_N) \in \mathcal{L}(\mathcal{X})^N$ is an N-tuple of the orthogonal projections P_k onto the subspaces $\mathcal{X}_k \subset \mathcal{X}$ such that $\mathcal{X} = \bigoplus_{k=1}^N \mathcal{X}_k$). Thus, we have $A_k = P_k A$, $B_k = P_k B$, $k = 1, \dots, N$, in the definition of FM system. The corresponding "N-D time-domain" equations have appeared first in the commutative case in [16, 17], and in the non-commutative case in [9].

2.4. **Structured, or Ball–Groenewald–Malakorn (BGM) systems.** Let us first recall briefly some auxiliary definitions from [9]. A bipartite graph G with the finite (ordered) sets of source vertices, S, of range vertices, R, and of edges, E, is called *admissible* if each pathwise-connected component G_k of G is a non-degenerate complete bipartite graph; i.e., its (non-empty) sets of source vertices, $S_k \subset S$, and of range vertices, $R_k \subset R$, are such that for each pair $(s, r) \in S_k \times R_k$ there is exactly one edge $e \in E$ with $s(e) = s$ and $r(e) = r$ (here $s(e)$ is the source vertex of the edge e, and $r(e)$ is the range vertex of the edge e).

Let $\{\mathcal{X}_s\}_{s\in S}$ and $\{\mathcal{X}_r\}_{r\in R}$, be two sets of Hilbert spaces indexed by the sets of the source vertices, S, and of the range vertices, R, respectively. We will assume that if the path components of the vertices $s_1, s_2 \in S$, $r_1, r_2 \in R$ are the same:

$$[s_1] = [s_2] = [r_1] = [r_2],$$

then

$$\mathcal{X}_{s_1} = \mathcal{X}_{s_2} = \mathcal{X}_{r_1} = \mathcal{X}_{r_2}.$$

In other words, a Hilbert space $\mathcal{X}_s$ or $\mathcal{X}_r$ is determined by the path-connected component of the vertex s or r.

With an admissible graph G and the collections of Hilbert spaces $\{\mathcal{X}_s\}_{s\in S}$ and $\{\mathcal{X}_r\}_{r\in R}$ specified, define a linear form $\Delta_\Sigma(z)$ in (commuting or non-commuting, depending on the setting) indeterminates $z = (z_e,\, e \in E)$ indexed by the edge set E of G, as follows. For each $e \in E$, define a matrix $I_{\Sigma,e} = [I_{\Sigma,e;s,r}]_{s\in S,\, r\in R}$ (with rows indexed by S and columns indexed by R) with block operator entries given by

$$I_{\Sigma,e;s,r} = \begin{cases} I_{\mathcal{X}_{[s(e)]}} = I_{\mathcal{X}_{[r(e)]}} & \text{if } (s,r) = (s(e), r(e)), \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

This matrix represents an operator from $\bigoplus_{r\in R} \mathcal{X}_r$ to $\bigoplus_{s\in S} \mathcal{X}_s$. Then set

$$\Delta_\Sigma(z) = \sum_{e\in E} I_{\Sigma,e} z_e. \tag{2.12}$$

(This linear form is denoted in [9] by $Z_G(z)$, however we prefer to change the notation here to avoid some confusion.)

BGM systems are a special case of FM systems, with $N = \#E$, $\mathcal{X} = \bigoplus_{s\in S} \mathcal{X}_{[s]}$ and with

$$U_z = U_z^{\mathrm{BGM}} = \begin{bmatrix} \Delta_\Sigma(z) & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2.13}$$

with a constant operator-block matrix

$$\begin{aligned} U &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{r,s}]_{r\in R,\, s\in S} & \mathrm{col}_{r\in R}[B_r] \\ \mathrm{row}_{s\in S}[C_s] & D \end{bmatrix} \\ &\in \mathcal{L}\left(\left(\bigoplus_{s\in S} \mathcal{X}_{[s]}\right) \oplus \mathcal{U}, \left(\bigoplus_{r\in R} \mathcal{X}_{[r]}\right) \oplus \mathcal{Y}\right). \end{aligned} \tag{2.14}$$

Thus, we have $A_k = I_{\Sigma,e_k} A,\ B_k = I_{\Sigma,e_k} B,\ k = 1, \ldots, N$, in the definition of FM system. The corresponding "N-D time-domain" equations have appeared first in the non-commutative case in [9].

Let us note that both a FM system and a GR system, in turn, can be viewed as special cases of a BGM system (see [9]). For the case of a FM system, the graph G consists of one component, $E = \{e_1, \ldots, e_N\},\ S = \{s_1\},\ R = \{r_1, \ldots, r_N\},\ A = \mathrm{col}_{k=1,\ldots,N}[A_k],\ B = \mathrm{col}_{k=1,\ldots,N}[B_k],\ \Delta_\Sigma(z) = \mathrm{row}_{k=1,\ldots,N}[z_k I_{\mathcal{X}}]$. For the case of a GR system, the graph G consists of N components, $E = \{e_1, \ldots, e_N\},\ S = \{s_1, \ldots, s_N\},\ R = \{r_1, \ldots, r_N\},\ A = [A_{kj}]_{k,j=1,\ldots,N},\ B = \mathrm{col}_{k=1,\ldots,N}[B_k],\ C = \mathrm{row}_{k=1,\ldots,N}[C_k],\ \Delta_\Sigma(z) = \mathrm{diag}_{k=1,\ldots,N}[z_k I_{\mathcal{X}_k}]$.

2.5. **Transfer function of a N-D system.** If A_z in (2.2) is linear homogeneous, i.e., $A_z = z\mathbf{A}$ (which is the case for all the types of systems above), then it follows from system equations (2.1) that

$$y(z) = T_\Sigma(z)u(z),$$

where

$$T_\Sigma(z) = D_z + \sum_{j=0}^{\infty} C_z A_z^j B_z$$

is a FPS (commutative or non-commutative, depending on the setting) with the coefficients in $\mathcal{L}(\mathcal{U},\mathcal{Y})$, which is called the *transfer function* of the N-D system Σ. Note that substitution of scalars $z_1,\ldots,z_N$ in the place of commuting indeterminates always turns T_{Σ^c} into a holomorphic function on some neighborhood of $0 \in \mathbb{C}^N$.

We will use a notation $\Sigma = (N; U_z; \mathcal{X},\mathcal{U},\mathcal{Y})$ for a N-D system (2.1). For a BGM system $\Sigma^{\rm BGM} = (N; U_z^{\rm BGM}; \mathcal{X},\mathcal{U},\mathcal{Y})$ we prefer to use notation $\Sigma^{\rm BGM} = (G; U; \mathcal{X},\mathcal{U},\mathcal{Y})$.

3. Dissipative and conservative N-D systems

3.1. **KV systems.** Recall (see [19]) that the commutative KV system $\Sigma^{{\rm KV},c} = (N; U_z^{\rm KV,c}; \mathcal{X},\mathcal{U},\mathcal{Y})$ is called *dissipative* (respectively, *conservative*) if for every $\zeta \in \mathbb{T}^N$ (the N-dimensional unit torus)

$$U_\zeta^{\rm KV} = \zeta\mathbf{U} := \sum_{k=1}^{N} \zeta_k U_k$$

is a contractive (respectively, unitary) operator, where $\mathbf{U} = (U_1,\ldots,U_N)$, with

$$U_k := \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \in \mathcal{L}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y}), \quad k = 1,\ldots,N.$$

Let $\mathcal{B}_N(\mathcal{U},\mathcal{Y})$ denote the class of commutative FPSs which become holomorphic contractive $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions on the unit polydisk $\mathbb{D}^N$, and $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$ its subclass consisting of FPSs F with $F_0 = 0$. Let $\mathcal{SA}_N(\mathcal{U},\mathcal{Y})$ denote the *Schur–Agler class* of commutative FPSs $F(z) = \sum_{t\in\mathbb{Z}_+^N} F_t z^t \in \mathcal{B}_N(\mathcal{U},\mathcal{Y})$ which satisfy $\|F(\mathbf{T})\| \le 1$ for any N-tuple $\mathbf{T} = (T_1,\ldots,T_N)$ of commuting strict contractions on some common Hilbert space $\mathcal{H}$, where

$$F(\mathbf{T}) := \sum_{t\in\mathbb{Z}_+^N} F_t \otimes \mathbf{T}^t,$$

and the series converges in the operator norm. Finally, let $\mathcal{SA}_N^0(\mathcal{U},\mathcal{Y})$ be a subclass of the class $\mathcal{SA}_N(\mathcal{U},\mathcal{Y})$ consisting of FPSs F with $F_0 = 0$. It was proved in [19] that an arbitrary dissipative commutative KV system $\Sigma^{{\rm KV},c} = (N; U_z^{\rm KV,c}; \mathcal{X},\mathcal{U},\mathcal{Y})$ has the transfer function $T_{\Sigma^{\rm KV,c}}$ in $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$. It was also proved in [19] that an arbitrary $F \in \mathcal{SA}_N^0(\mathcal{U},\mathcal{Y})$ has a *conservative KV-system realization*, i.e., a conservative KV system $\Sigma^{\rm KV,c} = (N; U_z^{\rm KV,c}; \mathcal{X},\mathcal{U},\mathcal{Y})$ such that $F = T_{\Sigma^{\rm KV,c}}$. In the cases $N = 1$ and $N = 2$ the classes $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$ and $\mathcal{SA}_N^0(\mathcal{U},\mathcal{Y})$ coincide, thus for these cases the class of transfer functions of dissipative commutative KV systems, as well as the class of transfer functions of conservative commutative KV systems, with the input space $\mathcal{U}$ and the output space $\mathcal{Y}$, is $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y}) = \mathcal{SA}_N^0(\mathcal{U},\mathcal{Y})$ (see Theorem 2.3 and Theorem 3.2 in [19]). It was shown in [22] that for every $N \ge 3$ these classes of transfer functions, in general, do not coincide. Namely, the class of transfer functions of dissipative commutative KV systems with some input space $\mathcal{U}$ and some output space $\mathcal{Y}$ is a strictly larger subclass in $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$ than $\mathcal{SA}_N^0(\mathcal{U},\mathcal{Y})$, while the latter, in case $\mathcal{U}=\mathcal{Y}$ coincides with the class of transfer functions of conservative commutative KV systems. However, it is still an open question as to exactly how large a class it is—in particular, whether it is a proper subclass in $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$ for some pair of $\mathcal{U}$ and $\mathcal{Y}$ or whether it coincides with all of $\mathcal{B}_N^0(\mathcal{U},\mathcal{Y})$ for every pair of $\mathcal{U}$ and $\mathcal{Y}$.

We shall call a non-commutative KV system $\Sigma^{\mathrm{KV,nc}}=(N;U_z^{\mathrm{KV,nc}};\mathcal{X},\mathcal{U},\mathcal{Y})$ *dissipative* (respectively, *conservative*) if for every $\mathbf{V}=(V_1,\ldots,V_N)\in\mathcal{U}^N$ (the set of N-tuples of unitary operators on a common Hilbert space, say $\mathcal{H}$),

$$\mathbf{U}\otimes\mathbf{V}:=\sum_{k=1}^N U_k\otimes V_k\in\mathcal{L}((\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H},(\mathcal{X}\oplus\mathcal{Y})\otimes\mathcal{H})$$

is a contractive (respectively, unitary) operator. Let us remark that we can consider the same collection of data $\Sigma^{\mathrm{KV}}=(N;U_z^{\mathrm{KV}};\mathcal{X},\mathcal{U},\mathcal{Y})$ for the commutative KV system, $\Sigma^{\mathrm{KV,c}}$, and for the non-commutative KV system, $\Sigma^{\mathrm{KV,nc}}$. Then (see [19, Proposition 2.4] and [24, Proposition 2.1]) $\Sigma^{\mathrm{KV,nc}}$ is conservative if and only if $\Sigma^{\mathrm{KV,c}}$ is conservative. It is obvious that if $\Sigma^{\mathrm{KV,nc}}$ is dissipative then $\Sigma^{\mathrm{KV,c}}$ is dissipative as well. The converse is true for $N=1$ and $N=2$, however is not true for $N\geq 3$. Indeed, for $N=1$ if $\|\zeta U\|\leq 1$ for every $\zeta\in\mathbb{T}$ then $\|U\|\leq 1$, and hence $\|U\otimes V\|\leq 1$ for every unitary operator V. For $N=2$ the generalized von Neumann inequality established in [4] implies that one has $\|\mathbf{U}\otimes\mathbf{T}\|\leq 1$ for every pair $\mathbf{T}=(T_1,T_2)$ of commuting contractions as soon as $\max_{\zeta\in\mathbb{T}^2}\|\zeta\mathbf{U}\|\leq 1$, therefore it is also true for every pair $\widetilde{\mathbf{T}}=(\widetilde{T_1},\widetilde{T_2})$ of non-commuting contractions. To show the latter, for a pair $\widetilde{\mathbf{T}}=(\widetilde{T_1},\widetilde{T_2})$ of non-commuting contractions acting on a Hilbert space $\mathcal{H}$ we define a pair of commuting contractions $\mathbf{T}=(T_1,T_2)$ acting on $\mathcal{H}\oplus\mathcal{H}$ by

$$T_k:=\begin{bmatrix}0&\widetilde{T_k}\\0&0\end{bmatrix},\qquad k=1,2,$$

(clearly, $T_kT_j=0,\ k,j=1,2$) and observe that

$$\|\mathbf{U}\otimes\widetilde{\mathbf{T}}\|=\|\mathbf{U}\otimes\mathbf{T}\|\leq 1.$$

In particular, for every pair $\mathbf{V}=(V_1,V_2)$ of unitary operators, one has $\|\mathbf{U}\otimes\mathbf{V}\|\leq 1$. For $N\geq 3$ it was deduced in [20, Theorem 5.5] from the main result of the paper [23] that there exists a commutative dissipative KV system $\Sigma^{\mathrm{KV,c}}$ such that $\|\mathbf{U}\otimes\mathbf{T}\|>1$ for some N-tuple $\mathbf{T}=(T_1,\ldots,T_N)$ of commuting contractions on a Hilbert space, say $\mathcal{H}$, thus the corresponding non-commutative KV system $\Sigma^{\mathrm{KV,nc}}$ cannot be dissipative. Indeed, by [29] $\mathbf{T}$ has a *unitary dilation*, i.e., an N-tuple $\mathbf{V}=(V_1,\ldots,V_N)$ of (not necessarily commuting) unitary operators on some Hilbert space $\mathcal{K}\supset\mathcal{H}$ such that

$$\mathbf{T}^w=P_{\mathcal{H}}\mathbf{V}^w|\mathcal{H},\qquad w\in\mathcal{F}_N,$$

thus

$$\begin{aligned}\|\mathbf{U}\otimes\mathbf{V}\| &= \left\|\sum_{k=1}^{N}U_k\otimes V_k\right\| \geq \left\|\sum_{k=1}^{N}(I_{\mathcal{X}\oplus\mathcal{Y}}\otimes P_{\mathcal{H}})(U_k\otimes V_k)|((\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H})\right\| \\ &= \left\|\sum_{k=1}^{N}U_k\otimes T_k\right\| = \|\mathbf{U}\otimes\mathbf{T}\| > 1.\end{aligned}$$

Recall (see [10]) that the *non-commutative Schur–Agler class* $\mathcal{SA}_N^{nc}(\mathcal{U},\mathcal{Y})$ consists of non-commutative FPSs $F(z)=\sum_{w\in\mathcal{F}_N}F_wz^w\in\mathcal{L}(\mathcal{U},\mathcal{Y})\left\langle\langle z_1,\ldots,z_N\rangle\right\rangle$ satisfying the condition that the series

$$F(\mathbf{T}):=\sum_{w\in\mathcal{F}_N}F_w\otimes\mathbf{T}^w\in\mathcal{L}(\mathcal{U}\otimes\mathcal{H},\mathcal{Y}\otimes\mathcal{H})$$

converges as a sequence of homogeneous polynomials in the norm topology to a contractive operator for every $\mathbf{T}\in\mathcal{D}^N$, i.e., for every N-tuple of strict contractions $\mathbf{T}=(T_1,\ldots,T_N)$ on a common Hilbert space $\mathcal{H}$. It was shown in [2] that for $F\in\mathcal{SA}_N^{nc}(\mathcal{U},\mathcal{Y})$ it suffices to verify the convergence and the property $\|F(\mathbf{T})\|\leq 1$ for all $\mathbf{T}$ from the *non-commutative matrix polydisk* $\mathcal{D}^N_{\rm matr}\subset\mathcal{D}^N$ (the latter is the disjoint union of the sets of all N-tuples of strictly contractive $n\times n$ matrices, $n=1,2,\ldots$). Let $\mathcal{SA}_N^{nc,0}(\mathcal{U},\mathcal{Y})$ denote the subclass of $\mathcal{SA}_N^{nc}(\mathcal{U},\mathcal{Y})$ consisting of FPSs F which satisfy $F_\emptyset=0$.

Proposition 3.1. *The transfer function $T_{\Sigma^{\rm KV,nc}}$ of a dissipative non-commutative KV system $\Sigma^{\rm KV,nc}=(N;U_z^{\rm KV,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ belongs to the class $\mathcal{SA}_N^{nc,0}(\mathcal{U},\mathcal{Y})$. An arbitrary $F\in\mathcal{SA}_N^{nc,0}(\mathcal{U},\mathcal{Y})$ admits a conservative KV realization, i.e., a conservative non-commutative KV system $\Sigma^{\rm KV,nc}=(N;U_z^{\rm KV,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ such that $F=T_{\Sigma^{\rm KV,nc}}$.*

Proof. Let $\Sigma^{\rm KV,nc}=(N;U_z^{\rm KV,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ be a dissipative non-commutative KV system. Let $n\in\mathbb{Z}_+\setminus\{0\}$ and $\mathbf{T}=(T_1,\ldots,T_N)\in\mathcal{D}^N_{\rm matr}\cap(\mathbb{C}^{n\times n})^N$. Then by the maximum principle

$$\|\mathbf{U}\otimes\mathbf{T}\|=\left\|\sum_{k=1}^{N}U_k\otimes T_k\right\|<1$$

(we recall that $\mathcal{U}^N\cap(\mathbb{C}^{n\times n})^N$ is the Shilov boundary of $\mathcal{D}^N_{\rm matr}\cap(\mathbb{C}^{n\times n})^N$ [27]). Hence,

$$\|\mathbf{A}\otimes\mathbf{T}\|=\|(P_{\mathcal{X}}\otimes I_n)(\mathbf{U}\otimes\mathbf{T})|(\mathcal{X}\otimes\mathbb{C}^n)\|<1.$$

This implies that

$$\begin{aligned}T_{\Sigma^{\rm KV,nc}}(\mathbf{T}) &= \mathbf{D}\otimes\mathbf{T}+(\mathbf{C}\otimes\mathbf{T})\sum_{j=0}^{\infty}(\mathbf{A}\otimes\mathbf{T})^j(\mathbf{B}\otimes\mathbf{T}) \\ &= \mathbf{D}\otimes\mathbf{T}+(\mathbf{C}\otimes\mathbf{T})(I_{\mathcal{X}\otimes\mathbb{C}^n}-\mathbf{A}\otimes\mathbf{T})^{-1}(\mathbf{B}\otimes\mathbf{T})\end{aligned}$$

is well defined. The 1D system $\Sigma_{\mathbf{T}}^{\rm KV,nc}:=(1;\mathbf{A}\otimes\mathbf{T},\mathbf{B}\otimes\mathbf{T},\mathbf{C}\otimes\mathbf{T},\mathbf{D}\otimes\mathbf{T};\mathcal{X}\otimes\mathbb{C}^n,\mathcal{U}\otimes\mathbb{C}^n,\mathcal{Y}\otimes\mathbb{C}^n)$ is dissipative, thus its transfer function

$$T_{\Sigma_{\mathbf{T}}^{\rm KV,nc}}(\lambda)=\lambda(\mathbf{D}\otimes\mathbf{T})+\lambda(\mathbf{C}\otimes\mathbf{T})(I_{\mathcal{X}\otimes\mathbb{C}^n}-\lambda(\mathbf{A}\otimes\mathbf{T}))^{-1}\lambda(\mathbf{B}\otimes\mathbf{T})$$

is a contractive holomorphic function on the unit disk $\mathbb{D}$. The function $T_{\Sigma^{\rm KV,nc}}(\mathbf{Z})$, as a function of matrix entries $(Z_k)_{ij}$, $k=1,\ldots,N$, $i,j=1,\ldots,n$, is holomorphic, and thus continuous on $\mathcal{D}^N_{\rm matr} \cap (\mathbb{C}^{n\times n})^N$. Therefore,

$$\|T_{\Sigma^{\rm KV,nc}}(\mathbf{T})\| = \lim_{\lambda\in\mathbb{D},\,\lambda\to 1}\|T_{\Sigma^{\rm KV,nc}}(\lambda\mathbf{T})\| = \lim_{\lambda\in\mathbb{D},\,\lambda\to 1}\|T_{\Sigma^{\rm KV,nc}_{\mathbf{T}}}(\lambda)\| \le 1,$$

which implies, together with the obvious property $(T_{\Sigma^{\rm KV,nc}})_\emptyset = 0$, that $T_{\Sigma^{\rm KV,nc}} \in \mathcal{SA}^{nc,0}_N(\mathcal{U},\mathcal{Y})$.

For the second statement of the Proposition, assume first that $F \in \mathcal{SA}^{nc,0}_N(\mathcal{Y})$, i.e., that $\mathcal{U}=\mathcal{Y}$. Then (see [24]) the FPS

$$f(z) := (I_\mathcal{Y} + F(z))(I_\mathcal{Y} - F(z))^{-1}$$

is well defined and belongs to the subclass $\mathcal{HA}^{nc,I}_N(\mathcal{Y})$ of the *non-commutative Herglotz–Agler class* $\mathcal{HA}^{nc}_N(\mathcal{Y})$, i.e., $f \in \mathcal{HA}^{nc}_N(\mathcal{Y})$ and $f_\emptyset = I_\mathcal{Y}$ (recall that the class $\mathcal{HA}^{nc}_N(\mathcal{Y})$ consists of non-commutative FPSs $\phi \in \mathcal{L}(\mathcal{Y})\langle\langle z_1,\dots,z_N\rangle\rangle$ such that the series $\phi(\mathbf{T}) = \sum_{w\in\mathcal{F}_N}\phi_w\otimes\mathbf{T}^w$ converges in the operator norm as a series of homogeneous polynomials, and Re $\phi(\mathbf{T}) \ge 0$ for every N-tuple $\mathbf{T}=(T_1,\dots,T_N)$ of strict contractions on a common Hilbert space). By [24, Theorem 3.1], there exists a conservative non-commutative KV system $\Sigma^{\rm KV,nc} = (N; U^{\rm KV,nc}_z; \mathcal{X},\mathcal{Y},\mathcal{Y})$ such that

$$f(z) = P_\mathcal{Y}(I_{\mathcal{X}\oplus\mathcal{Y}} + z\mathbf{U})(I_{\mathcal{X}\oplus\mathcal{Y}} - z\mathbf{U})^{-1}|\mathcal{Y}. \tag{3.1}$$

Therefore,

$$\begin{aligned} F(z) &= (f(z)-I_\mathcal{Y})(f(z)+I_\mathcal{Y})^{-1}\\ &= I_\mathcal{Y} - 2(f(z)+I_\mathcal{Y})^{-1}\\ &= I_\mathcal{Y} - 2(P_\mathcal{Y}(I_{\mathcal{X}\oplus\mathcal{Y}} + z\mathbf{U})(I_{\mathcal{X}\oplus\mathcal{Y}} - z\mathbf{U})^{-1}|\mathcal{Y} + I_\mathcal{Y})^{-1}\\ &= I_\mathcal{Y} - (P_\mathcal{Y}(I_{\mathcal{X}\oplus\mathcal{Y}} - z\mathbf{U})^{-1}|\mathcal{Y})^{-1}\\ &= I_\mathcal{Y} - (I_\mathcal{Y} - z\mathbf{D} - z\mathbf{C}(I_\mathcal{X} - z\mathbf{A})^{-1}z\mathbf{B})\\ &= z\mathbf{D} + z\mathbf{C}(I_\mathcal{X} - z\mathbf{A})^{-1}z\mathbf{B}\\ &= T_{\Sigma^{KV,nc}}(z) \end{aligned}$$

(we used here the well known Schur complement formulas).

Consider now the general case $F \in \mathcal{SA}^{nc,0}_N(\mathcal{U},\mathcal{Y})$. Define

$$\widetilde{F}(z) := \begin{bmatrix} 0 & 0\\ F(z) & 0\end{bmatrix} \in \mathcal{L}(\mathcal{U}\oplus\mathcal{Y})\langle\langle z_1,\dots,z_N\rangle\rangle.$$

Clearly, $\widetilde{F} \in \mathcal{SA}^{nc,0}_N(\mathcal{U}\oplus\mathcal{Y})$. By the result of the previous paragraph, there is a conservative non-commutative KV system $\widetilde{\Sigma}^{\rm KV,nc} = (N; \widetilde{U}^{\rm KV,nc}_z; \widetilde{\mathcal{X}},\mathcal{U}\oplus\mathcal{Y},\mathcal{U}\oplus\mathcal{Y})$ such that $T_{\widetilde{\Sigma}^{\rm KV,nc}} = \widetilde{F}$. In particular,

$$\widetilde{D_k} = \begin{bmatrix} 0 & 0\\ D_k & 0\end{bmatrix} \in \mathcal{L}(\mathcal{U}\oplus\mathcal{Y})$$

where we set $D_k := F_{g_k},\ k=1,\dots,N$. We may also write

$$\widetilde{B_k} = \begin{bmatrix} \widetilde{B_k}^\mathcal{U} & \widetilde{B_k}^\mathcal{Y}\end{bmatrix} \in \mathcal{L}(\mathcal{U}\oplus\mathcal{Y},\widetilde{\mathcal{X}}),\quad k=1,\dots,N,$$

$$\widetilde{C_k} = \begin{bmatrix} \widetilde{C_k}^\mathcal{U}\\ \widetilde{C_k}^\mathcal{Y}\end{bmatrix} \in \mathcal{L}(\widetilde{\mathcal{X}},\mathcal{U}\oplus\mathcal{Y}),\quad k=1,\dots,N.$$

We define another non-commutative KV system $\Sigma^{\rm KV,nc} = (N; U_z^{\rm KV,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ as follows. Set $\mathcal{X} := \cdots \oplus \mathcal{U} \oplus \mathcal{U} \oplus \widetilde{\mathcal{X}} \oplus \mathcal{Y} \oplus \mathcal{Y} \oplus \cdots$, and with respect to this orthogonal sum decomposition,

$$A_k := \begin{bmatrix} \ddots & & & & & \\ & \delta_{1k} & & & & \\ & & \widetilde{C}_k^{\mathcal{U}} & & & \\ & & \boxed{\widetilde{A}_k} & \widetilde{B}_k^{\mathcal{Y}} & & \\ & & & & \delta_{1k} & \\ & & & & & \ddots \end{bmatrix}, \quad B_k := \begin{bmatrix} \vdots \\ 0 \\ \boxed{\widetilde{B}_k^{\mathcal{U}}} \\ 0 \\ \vdots \end{bmatrix},$$

$$C_k := \begin{bmatrix} \dots & 0 & \boxed{\widetilde{C}_k^{\mathcal{Y}}} & 0 & \dots \end{bmatrix}, \quad k = 1, \dots, N.$$

Here δ_{1k} is the Kronecker delta, blank places in the matrix A_k correspond to zero operator blocks, and the frames allocate an $\mathcal{L}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}})$-block, an $\mathcal{L}(\mathcal{U}, \widetilde{\mathcal{X}})$-block, an $\mathcal{L}(\widetilde{\mathcal{X}}, \mathcal{Y})$-block in matrices A_k, B_k, and C_k, respectively. It is an easy exercise to check that the just defined KV system $\Sigma^{\rm KV,nc}$ is conservative, and that $T_{\Sigma^{\rm KV,nc}} = F$. (Hint: observe that $F(z) = z\mathbf{D} + z\widetilde{\mathbf{C}}^{\mathcal{Y}}(I_{\widetilde{\mathcal{X}}} - z\widetilde{\mathbf{A}})^{-1} z\widetilde{\mathbf{B}}^{\mathcal{U}}$.) □

Since any conservative non-commutative KV system is dissipative, this theorem means that the class of transfer functions of dissipative non-commutative KV systems, as well as the class of transfer functions of conservative non-commutative KV systems, with the input space $\mathcal{U}$ and the output space $\mathcal{Y}$, is $\mathcal{SA}_N^{nc,0}(\mathcal{U}, \mathcal{Y})$.

Remark 3.2. For the proof of the first statement of Proposition 3.1 we used here the same argument as for the proof of its commutative counterpart, [19, Theorem 2.3]. The proof of the second statement goes through without changes in the commutative setting for functions/commutative formal power series from the class $\mathcal{SA}_N^0(\mathcal{U}, \mathcal{Y})$ by making use of Agler's theorem on representation of functions f from the *Herglotz–Agler class* $\mathcal{HA}_N(\mathcal{U}, \mathcal{Y})$ with an additional condition $f(0) = I_{\mathcal{Y}}$ [1, Theorem 1.8] instead of of its non-commutative analogue (see [24, Theorem 3.1]). In the commutative case, this proof is new and becomes the third proof of this result (the first proof appeared in [19] and used Agler's identities and the "lurking isometry" argument with some extra observation connected with the condition $F(0) = 0$; the second proof follows from the identifications between commutative conservative KV, FM and GR systems established in [11, Remark 2.5]).

3.2. **FM systems (the polydisk case).** Let us note that the properties of dissipativity and conservativity of KV systems in Section 3.1 were associated with a certain domain. Namely, with the unit polydisk ($\mathbb{D}^N$ in the commutative setting, and $\mathcal{D}^N$ in the non-commutative setting), or with its essential boundary ($\mathbb{T}^N$ in the commutative setting, and $\mathcal{U}^N$ in the non-commutative setting - in this case the notion of "essential boundary" is not defined rigorously, and we use it as slang). More precisely, these properties were defined via values of U_z on $\mathbb{T}^N$ ($\mathcal{U}^N$), and then the transfer functions became well defined and holomorphic on the (commutative or non-commutative, depending on the setting) unit polydisk. In this section we study dissipative and conservative FM systems in the polydisk version.

A commutative FM system $\Sigma^{\mathrm{FM,c}} = (N; U_z^{\mathrm{FM,c}}; \mathcal{X},\mathcal{U},\mathcal{Y})$ will be called $\mathbb{D}^N$-*dissipative* (respectively, $\mathbb{D}^N$*-conservative*) if for every $\zeta \in \mathbb{T}^N$,

$$U_\zeta^{\mathrm{FM}} = \begin{bmatrix} \zeta\mathbf{A} & \zeta\mathbf{B} \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$$

is a contractive (respectively, unitary) operator.

Proposition 3.3. *The transfer function $T_{\Sigma^{\mathrm{FM,c}}}$ of a $\mathbb{D}^N$-dissipative commutative FM system $\Sigma^{\mathrm{FM,c}} = (N; U_z^{\mathrm{FM,c}}; \mathcal{X},\mathcal{U},\mathcal{Y})$ belongs to the class $\mathcal{B}_N(\mathcal{U},\mathcal{Y})$.*

Proof. Since by the maximum principle $U_z \in \mathcal{B}_N(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$, one has also $A_z = z\mathbf{A} = P_{\mathcal{X}}U_z|\mathcal{X} \in \mathcal{B}_N(\mathcal{X})$. This implies that $T_{\Sigma^{\mathrm{FM,c}}}(z) = D + C(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B}$ is well defined and holomorphic on $\mathbb{D}^N$. Fix an arbitrary $z^0 \in \mathbb{D}^N$. Then the 1D system $\Sigma_{z^0}^{\mathrm{FM,c}} := (1; z^0\mathbf{A}, z^0\mathbf{B}, C, D; \mathcal{X},\mathcal{U},\mathcal{Y})$ is dissipative, thus its transfer function $T_{\Sigma_{z^0}^{\mathrm{FM,c}}}(\lambda) = D + C(I_{\mathcal{X}} - \lambda z^0\mathbf{A})^{-1}\lambda z^0\mathbf{B}$ is a contractive holomorphic function on the unit disk $\mathbb{D}$. The function $T_{\Sigma^{\mathrm{FM,c}}}(z)$ is holomorphic, and thus continuous, at z^0. Therefore,

$$\|T_{\Sigma^{\mathrm{FM,c}}}(z^0)\| = \lim_{\lambda\in\mathbb{D},\,\lambda\to 1} \|T_{\Sigma^{\mathrm{FM,c}}}(\lambda z^0)\| = \lim_{\lambda\in\mathbb{D},\,\lambda\to 1} \|T_{\Sigma_{z^0}^{\mathrm{FM,c}}}(\lambda)\| \le 1.$$

This completes the proof. □

It has been shown implicitly in [11] that an arbitrary commutative formal power series F from the *Schur–Agler* class $\mathcal{SA}_N(\mathcal{U},\mathcal{Y})$ has a $\mathbb{D}^N$*-conservative FM system realization* $\Sigma^{\mathrm{FM,c}} = (N; U_z^{\mathrm{FM,c}}; \mathcal{X},\mathcal{U},\mathcal{Y})$, i.e., $F = T_{\Sigma^{\mathrm{FM,c}}}$. Indeed, by Agler's realization theorem (see [1]), F has a *conservative commutative GR system realization* $\Sigma^{\mathrm{GR,c}} = (N; U_z^{\mathrm{GR,c}}; \mathcal{X},\mathcal{U},\mathcal{Y})$, with the linear function U_z^{GR} as in (2.9), where the conservativity means that the matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

in (2.9) is unitary. It is straightforward to see that a commutative GR system is conservative if and only if this system viewed as a FM system is $\mathbb{D}^N$-conservative, since the linear function $z\mathbf{P}$ in (2.9) is unitary on $\mathbb{T}^N$. Thus, Agler's theorem guarantees the existence of a $\mathbb{D}^N$-conservative FM system realization for $F \in \mathcal{SA}_N(\mathcal{U},\mathcal{Y})$. In the cases $N = 1$ and $N = 2$ the classes $\mathcal{B}_N(\mathcal{U},\mathcal{Y})$ and $\mathcal{SA}_N(\mathcal{U},\mathcal{Y})$ coincide, thus for these cases the class of transfer functions of $\mathbb{D}^N$-dissipative FM systems, as well as the class of transfer functions of $\mathbb{D}^N$-conservative FM systems, with the input space $\mathcal{U}$ and the output space $\mathcal{Y}$, is $\mathcal{B}_N(\mathcal{U},\mathcal{Y}) = \mathcal{SA}_N(\mathcal{U},\mathcal{Y})$. Let us show that for every $N \ge 3$ these classes of transfer functions, in general, do not coincide. By [23], for every $N \ge 3$ there exists a (finite-dimensional) Hilbert space $\mathcal{X}$ and a linear function $B_z = z\mathbf{B} \in \mathcal{L}(\mathcal{X})[z_1,\ldots,z_N]$ such that $B_z \in \mathcal{B}_N(\mathcal{X}) \setminus \mathcal{SA}_N(\mathcal{X})$. Then the system $\Sigma^{\mathrm{FM,c}} := (N; U_z^{\mathrm{FM,c}} = \begin{bmatrix} 0 & B_z \\ I_{\mathcal{X}} & 0 \end{bmatrix}; \mathcal{X},\mathcal{X},\mathcal{X})$ is $\mathbb{D}^N$-dissipative, and its transfer function $T_{\Sigma^{\mathrm{FM,c}}} = B_z \in \mathcal{B}_N(\mathcal{X}) \setminus \mathcal{SA}_N(\mathcal{X})$. Thus, the class of transfer functions of $\mathbb{D}^N$-dissipative FM systems with the input and output spaces equal to $\mathcal{X}$ is larger than $\mathcal{SA}_N(\mathcal{X})$, while the latter coincides with the class of transfer functions of $\mathbb{D}^N$-conservative FM systems with the input and output spaces equal to $\mathcal{X}$. As in the case of commutative KV systems, it is still an open question as to exactly how large a class it is—in particular, whether the class of transfer functions of $\mathbb{D}^N$-dissipative FM systems with the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ is a proper subclass in $\mathcal{B}_N(\mathcal{U},\mathcal{Y})$ for some pair of spaces $\mathcal{U}$ and $\mathcal{Y}$ or whether this class coincides with all of $\mathcal{B}_N(\mathcal{U},\mathcal{Y})$ for every pair of $\mathcal{U}$ and $\mathcal{Y}$.

A non-commutative FM system $\Sigma^{\rm FM,nc} = (N; U_z^{\rm FM,nc}; \mathcal{X},\mathcal{U},\mathcal{Y})$ will be called $\mathcal{D}^N$*-dissipative* (respectively, $\mathcal{D}^N$*-conservative*) if for every $\mathbf{V}\in\mathcal{U}^N\cap\mathcal{L}(\mathcal{H})^N$, with some Hilbert space $\mathcal{H}$,

$$U_{\mathbf{V}}^{\rm FM} := \begin{bmatrix} \mathbf{A}\otimes\mathbf{V} & \mathbf{B}\otimes\mathbf{V} \\ C\otimes I_{\mathcal{H}} & D\otimes I_{\mathcal{H}} \end{bmatrix} \in \mathcal{L}((\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}, (\mathcal{X}\oplus\mathcal{Y})\otimes\mathcal{H})$$

is a contractive (respectively, unitary) operator.

Note that the non-commutative FM system $\Sigma^{\rm FM,nc} = (N; U_z^{\rm FM,nc}; \mathcal{X},\mathcal{U},\mathcal{Y})$ is $\mathcal{D}^N$-conservative if and only if the corresponding commutative FM system $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X},\mathcal{U},\mathcal{Y})$ is $\mathbb{D}^N$-conservative. Indeed, the "only if" part is obvious. For the proof of the "if" part, suppose that $\Sigma^{\rm FM,c}$ is $\mathbb{D}^N$-conservative. Then for each $\zeta\in\mathbb{T}^N$ the operator

$$U_\zeta^{\rm FM} = \begin{bmatrix} \zeta\mathbf{A} & \zeta\mathbf{B} \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$$

is unitary, i.e., isometric and coisometric. The isometry relations for $U_\zeta^{\rm FM}$, $\zeta\in\mathbb{T}^N$, altogether are equivalent to the following equalities (see (2.16) in [11]):

$$\sum_{k=1}^N A_k^*A_k + C^*C = I_{\mathcal{X}}, \quad \textstyle\sum_{k=1}^N A_k^*B_k + C^*D = 0, \quad \displaystyle\sum_{k=1}^N B_k^*B_k + D^*D = I_{\mathcal{U}},$$
$$A_k^*A_j = 0, \qquad A_k^*B_j = 0, \qquad B_k^*B_j = 0 \quad \text{for } k\neq j. \tag{3.2}$$

For an arbitrary N-tuple $\mathbf{V}=(V_1,\dots,V_N)$ of (not necessarily commuting) unitaries on a common Hilbert space $\mathcal{H}$ by virtue of (3.2) we obtain

$$\begin{aligned} U_{\mathbf{V}}^{\rm FM*}U_{\mathbf{V}}^{\rm FM} &= \begin{bmatrix} \mathbf{A}\otimes\mathbf{V} & \mathbf{B}\otimes\mathbf{V} \\ C\otimes I_{\mathcal{H}} & D\otimes I_{\mathcal{H}} \end{bmatrix}^* \begin{bmatrix} \mathbf{A}\otimes\mathbf{V} & \mathbf{B}\otimes\mathbf{V} \\ C\otimes I_{\mathcal{H}} & D\otimes I_{\mathcal{H}} \end{bmatrix} \\ &= \begin{bmatrix} \left(\sum_{k=1}^N A_k^*A_k + C^*C\right)\otimes I_{\mathcal{H}} & 0 \\ 0 & \left(\sum_{k=1}^N B_k^*B_k + D^*D\right)\otimes I_{\mathcal{H}} \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{X}\otimes\mathcal{H}} & 0 \\ 0 & I_{\mathcal{U}\otimes\mathcal{H}} \end{bmatrix}, \end{aligned}$$

i.e., $U_{\mathbf{V}}^{\rm FM}$ is an isometry. Analogously, from the coisometry relations for $U_\zeta^{\rm FM}$, $\zeta\in\mathbb{T}^N$, one obtains the coisometry relations for $U_{\mathbf{V}}^{\rm FM}$, $\mathbf{V}\in\mathcal{U}^N$. Thus $U_{\mathbf{V}}^{\rm FM}$ is unitary for every $\mathbf{V}\in\mathcal{U}^N$, i.e., the system $\Sigma^{\rm FM,nc}$ is $\mathcal{D}^N$-conservative.

It is clear that if $\Sigma^{{\rm FM},nc}$ is $\mathcal{D}^N$-dissipative then $\Sigma^{{\rm FM},c}$ is $\mathbb{D}^N$-dissipative. The converse is true in the cases $N=1$ and $N=2$ (this can be shown in the same way as in the case of KV systems from the previous subsection), and is not true in the case $N\geq 3$. Indeed, in the case $N\geq 3$, by [23] there exists an N-tuple $\mathbf{A}=(A_1,\dots,A_N)$ of operators on some finite-dimensional Hilbert space $\mathcal{X}$ and an N-tuple $\mathbf{T}=(T_1,\dots,T_N)$ of contractions on some other finite-dimensional Hilbert space $\mathcal{H}$ such that $\|\mathbf{A}\otimes\mathbf{T}\| > 1 = \max_{\zeta\in\mathbb{T}^N}\|\zeta\mathbf{A}\|$. Then the commutative FM system $\Sigma^{{\rm FM},c} := (N; U_z^{\rm FM,c} = z\mathbf{A}; \mathcal{X}, \{0\}, \{0\})$ is $\mathbb{D}^N$-dissipative, and the corresponding non-commutative FM system $\Sigma^{{\rm FM},nc} := (N; U_z^{\rm FM,nc} = z\mathbf{A}; \mathcal{X}, \{0\}, \{0\})$ is not $\mathcal{D}^N$-dissipative for the same unitary dilation reasoning as in the previous subsection for a KV system.

Proposition 3.4. *The transfer function* $T_{\Sigma^{\mathrm{FM},nc}}$ *of a* $\mathcal{D}^N$*-dissipative non-commutative FM system* $\Sigma^{\mathrm{FM},nc} = (N; U_z^{\mathrm{FM,nc}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *belongs to the class* $\mathcal{SA}_N^{nc}(\mathcal{U}, \mathcal{Y})$*. An arbitrary* $F \in \mathcal{SA}_N^{nc}(\mathcal{U}, \mathcal{Y})$ *admits a* $\mathcal{D}^N$*-conservative realization, i.e., a* $\mathcal{D}^N$*-conservative non-commutative FM system* $\Sigma^{\mathrm{FM,nc}} = (N; U_z^{\mathrm{FM,nc}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *such that* $F = T_{\Sigma^{\mathrm{FM,nc}}}$*.*

Proof. Let $\Sigma^{\mathrm{FM},nc} = (N; U_z^{\mathrm{FM,nc}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be $\mathcal{D}^N$-dissipative. Let $n \in \mathbb{Z}_+ \setminus \{0\}$ and $\mathbf{T} = (T_1, \ldots, T_N) \in \mathcal{D}^N_{\mathrm{matr}} \cap (\mathbb{C}^{n\times n})^N$. Then by the maximum principle

$$\|U_{\mathbf{T}}^{\mathrm{FM}}\| = \left\| \begin{bmatrix} \mathbf{A} \otimes \mathbf{T} & \mathbf{B} \otimes \mathbf{T} \\ C \otimes I_n & D \otimes I_n \end{bmatrix} \right\| \le 1$$

(we recall that $\mathcal{U}^N \cap (\mathbb{C}^{n\times n})^N$ is the essential (or Shilov) boundary of $\mathcal{D}^N_{\mathrm{matr}} \cap (\mathbb{C}^{n\times n})^N$ [27]). Hence, due to the fact that $\mathbf{T} = (T_1, \ldots, T_N)$ is an N-tuple of strict contractions,

$$\|\mathbf{A} \otimes \mathbf{T}\| = \|(P_{\mathcal{X}} \otimes I_n) U_{\mathbf{T}}^{\mathrm{FM}} |(\mathcal{X} \otimes \mathbb{C}^n)\| < 1.$$

This implies that

$$\begin{aligned} T_{\Sigma^{\mathrm{FM},nc}}(\mathbf{T}) &= D \otimes I_n + (C \otimes I_n) \sum_{j=0}^{\infty} (\mathbf{A} \otimes \mathbf{T})^j (\mathbf{B} \otimes \mathbf{T}) \\ &= D \otimes I_n + (C \otimes I_n)(I_{\mathcal{X}\otimes\mathbb{C}^n} - \mathbf{A} \otimes \mathbf{T})^{-1} (\mathbf{B} \otimes \mathbf{T}) \end{aligned}$$

is well defined. The 1D system $\Sigma_{\mathbf{T}}^{\mathrm{FM,nc}} := (1; \mathbf{A} \otimes \mathbf{T}, \mathbf{B} \otimes \mathbf{T}, C \otimes I_n, D \otimes I_n; \mathcal{X} \otimes \mathbb{C}^n, \mathcal{U} \otimes \mathbb{C}^n, \mathcal{Y} \otimes \mathbb{C}^n)$ is dissipative, thus its transfer function

$$T_{\Sigma_{\mathbf{T}}^{\mathrm{FM,nc}}}(\lambda) = D \otimes I_n + (C \otimes I_n)(I_{\mathcal{X}\otimes\mathbb{C}^n} - \lambda(\mathbf{A} \otimes \mathbf{T}))^{-1} \lambda(\mathbf{B} \otimes \mathbf{T})$$

is a contractive holomorphic function on the unit disk $\mathbb{D}$. The function $T_{\Sigma^{\mathrm{FM,nc}}}(\mathbf{Z})$, as a function of matrix entries $(Z_k)_{ij}$, $k = 1, \ldots, N$, $i, j = 1, \ldots, n$, is holomorphic, and thus continuous on $\mathcal{D}^N_{\mathrm{matr}} \cap (\mathbb{C}^{n\times n})^N$. Therefore,

$$\|T_{\Sigma^{\mathrm{FM,nc}}}(\mathbf{T})\| = \lim_{\lambda\in\mathbb{D},\, \lambda\to 1} \|T_{\Sigma^{\mathrm{FM,nc}}}(\lambda\mathbf{T})\| = \lim_{\lambda\in\mathbb{D},\, \lambda\to 1} \|T_{\Sigma_{\mathbf{T}}^{\mathrm{FM,nc}}}(\lambda)\| \le 1,$$

which implies that $T_{\Sigma^{\mathrm{FM,nc}}} \in \mathcal{SA}_N^{nc}(\mathcal{U}, \mathcal{Y})$.

It is straightforward to see that a non-commutative GR system is conservative (which means that

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

in (2.9) is unitary) if and only if this system viewed as a non-commutative FM system is $\mathcal{D}^N$-conservative, since the linear function $z\mathbf{P}$ in (2.9) is unitary on $\mathcal{U}^N$. By [10], an arbitrary $F \in \mathcal{SA}_N^{nc}(\mathcal{U}, \mathcal{Y})$ admits a conservative non-commutative GR system realization, which can be viewed as a $\mathcal{D}^N$-conservative non-commutative FM system realization. □

Since any $\mathcal{D}^N$-conservative non-commutative FM system is $\mathcal{D}^N$-dissipative, we obtain the following.

Corollary 3.5. *The class of transfer functions of* $\mathcal{D}^N$*-dissipative non-commutative FM systems, as well as the one of* $\mathcal{D}^N$*-conservative non-commutative FM systems, with the input space* $\mathcal{U}$ *and the output space* $\mathcal{Y}$*, is* $\mathcal{SA}_N^{nc}(\mathcal{U}, \mathcal{Y})$*.*

3.3. **BGM systems.** We will call the (commutative or non-commutative) BGM system $\Sigma^{\rm BGM} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *dissipative* (respectively, *conservative*) if the operator $U \in \mathcal{L}\left(\left(\bigoplus_{s\in S} \mathcal{X}_s\right) \oplus \mathcal{U}, \left(\bigoplus_{r\in R} \mathcal{X}_r\right) \oplus \mathcal{Y}\right)$ is contractive (respectively, unitary). The natural domain associated with a commutative BGM system $\Sigma = \Sigma^{\rm BGM,c}$ is

$$\mathbb{D}_{\Delta_\Sigma} := \{z \in \mathbb{C}^N : \ \|\Delta_\Sigma(z)\| < 1\},$$

where Δ_Σ is defined by (2.12). Let us remark that in the case of FM systems considered as a special case of BGM systems (see the last paragraph in Section 2.4), this domain becomes the unit ball:

$$\mathbb{D}_{\Delta_\Sigma} = \mathbb{B}^N := \{z \in \mathbb{C}^N : \ |z_1|^2 + \cdots + |z_N|^2 < 1\},$$

and in the case of GR systems considered as a special case of BGM systems (see ibidem), this domain becomes the unit polydisk: $\mathbb{D}_{\Delta_\Sigma} = \mathbb{D}^N$. The domain $\mathbb{D}_{\Delta_\Sigma}$ is a special case of polynomially defined domains (see [3, 8]) where the polynomial is a linear function Δ_Σ. The *Schur–Agler class associated with the domain* $\mathbb{D}_{\Delta_\Sigma}$ (see [3, 8] for the definition of the more general Schur–Agler class associated with a domain with polynomial rather than linear defining function) is the class $\mathcal{SA}_{\Delta_\Sigma}(\mathcal{U}, \mathcal{Y})$ of commutative formal power series

$$F(z) = \sum_{t\in\mathbb{Z}_+^N} F_t z^t \in \mathcal{L}(\mathcal{U}, \mathcal{Y})[[z_1, \ldots, z_N]]$$

which become holomorphic functions on $\mathbb{D}_{\Delta_\Sigma}$ and satisfy the condition that

$$F(\mathbf{T}) := \sum_{t\in\mathbb{Z}_+^N} F_t \otimes \mathbf{T}^t$$

is a contraction for any N-tuple $\mathbf{T} = (T_1, \ldots, T_N)$ of commuting bounded linear operators on a common Hilbert space subject to the condition

$$\|\Delta_\Sigma(\mathbf{T})\| = \left\| \sum_{k=1}^N I_{\Sigma,e_k} \otimes T_k \right\| < 1. \tag{3.3}$$

It is easy to show that the transfer function $T_{\Sigma^{\rm BGM,c}}$ of a commutative dissipative BGM system $\Sigma^{\rm BGM,c} = (G, U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ belongs to the class $\mathcal{SA}_{\Delta_\Sigma}(\mathcal{U}, \mathcal{Y})$ (one can use the same closed-loop-transformation argument as was used for the proof of implication (5)$\Longrightarrow$(1) in [8, Theorem 1.5]). On the other hand, it follows from [8, Theorem 1.5] that every $F \in \mathcal{SA}_{\Delta_\Sigma}(\mathcal{U}, \mathcal{Y})$ admits a conservative commutative BGM-system realization. Thus, we obtain the following result.

Proposition 3.6. *The class of transfer functions of commutative dissipative BGM systems (as well as the corresponding class for commutative conservative BGM systems) with the input space* $\mathcal{U}$ *and the output space* $\mathcal{Y}$ *coincides with* $\mathcal{SA}_{\Delta_\Sigma}(\mathcal{U}, \mathcal{Y})$.

The non-commutative domain $\mathcal{D}_{\Delta_\Sigma}$ associated with a non-commutative BGM system $\Sigma = \Sigma^{\rm BGM,nc}$ consists of N-tuples $\mathbf{T} = (T_1, \ldots, T_N)$ of (not necessarily commuting) bounded linear operators on a common separable Hilbert space subject to the condition

$$\|\Delta_\Sigma(\mathbf{T})\| = \left\| \sum_{k=1}^N I_{\Sigma,e_k} \otimes T_k \right\| < 1. \tag{3.4}$$

Let us remark that in the case of FM system considered as a special case of BGM system (see the last paragraph in Section 2.4), this domain becomes the non-commutative unit ball: $\mathcal{D}_{\Delta_\Sigma}=\mathcal{B}^N$, i.e. consists of *strict row contractions*

$$\mathbf{T}=(T_1,\dots,T_N):\ \sum_{k=1}^N T_kT_k^*<1,$$

and in the case of GR system considered as a special case of BGM system (see ibidem), this domain becomes the non-commutative unit polydisk: $\mathcal{D}_{\Delta_\Sigma}=\mathcal{D}^N$. The *non-commutative Schur–Agler class associated with the domain* $\mathcal{D}_{\Delta_\Sigma}$ (see [10]) is the class $\mathcal{SA}^{nc}_{\Delta_\Sigma}(\mathcal{U},\mathcal{Y})$ of non-commutative formal power series

$$F(z)=\sum_{w\in\mathcal{F}_N}F_wz^w\in\mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z_1,\dots,z_N\rangle\rangle$$

which satisfy the condition that the series

$$F(\mathbf{T}):=\sum_{w\in\mathcal{F}_N}F_w\otimes\mathbf{T}^w$$

converges as a sequence of homogeneous polynomials in the operator norm to a contraction for any $\mathbf{T}\in\mathcal{D}_{\Delta_\Sigma}$. It was shown in [10] that the transfer function $T_{\Sigma^{\mathrm{BGM,nc}}}$ of a non-commutative dissipative BGM system $\Sigma^{\mathrm{BGM,nc}}=(G,U;\mathcal{X},\mathcal{U},\mathcal{Y})$ belongs to the class $\mathcal{SA}^{nc}_{\Delta_\Sigma}(\mathcal{U},\mathcal{Y})$, and every $F\in\mathcal{SA}^{nc}_{\Delta_\Sigma}(\mathcal{U},\mathcal{Y})$ admits a conservative non-commutative BGM-system realization. Thus, the class of transfer functions of non-commutative dissipative BGM systems (as well as the corresponding class for non-commutative conservative BGM systems) with the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ coincides with $\mathcal{SA}^{nc}_{\Delta_\Sigma}(\mathcal{U},\mathcal{Y})$.

Remark 3.7. In order to get $F\in\mathcal{SA}^{nc}_{\Delta_\Sigma}(\mathcal{U},\mathcal{Y})$ it suffices to have the inequality $\|F(\mathbf{T})\|\le 1$ valid for every N-tuple $\mathbf{T}$ of finite-dimensional operators ($n\times n$ matrices, $n=1,2,\dots$) subject to condition (3.4) (in this case we will write $\mathbf{T}\in\mathcal{D}_{\Delta_\Sigma,\mathrm{matr}}\subset\mathcal{D}_{\Delta_\Sigma}$). This fact has been proved in [2] for the case where $\mathcal{D}_{\Delta_\Sigma}=\mathcal{D}^N$ (see the paragraph preceding Proposition 3.1), however the proof extends verbatim (with some obvious changes in notations) to an arbitrary non-commutative domain $\mathcal{D}_{\Delta_\Sigma}$.

4. Dilations of KV systems

4.1. **The non-commutative setting.** We will say that the non-commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,nc}}=(N;\widetilde{U}_z^{\mathrm{KV,nc}};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a *dilation of the non-commutative KV system* $\Sigma^{\mathrm{KV,nc}}=(N;U_z^{\mathrm{KV,nc}};\mathcal{X},\mathcal{U},\mathcal{Y})$ if for every $n\in\mathbb{Z}_+\backslash\{0\}$ and $\mathbf{Z}=(Z_1,\dots,Z_N)\in(\mathbb{C}^{n\times n})^N$ the 1D system $\widetilde{\Sigma}^{\mathrm{KV,nc}}_{\mathbf{Z}}:=(1;\widetilde{\mathbf{A}}\otimes\mathbf{Z},\widetilde{\mathbf{B}}\otimes\mathbf{Z},\widetilde{\mathbf{C}}\otimes\mathbf{Z},\widetilde{\mathbf{D}}\otimes\mathbf{Z};\widetilde{\mathcal{X}}\otimes\mathbb{C}^n,\mathcal{U}\otimes\mathbb{C}^n,\mathcal{Y}\otimes\mathbb{C}^n)$ is a dilation of the 1D system $\Sigma^{\mathrm{KV,nc}}_{\mathbf{Z}}:=(1;\mathbf{A}\otimes\mathbf{Z},\mathbf{B}\otimes\mathbf{Z},\mathbf{C}\otimes\mathbf{Z},\mathbf{D}\otimes\mathbf{Z};\mathcal{X}\otimes\mathbb{C}^n,\mathcal{U}\otimes\mathbb{C}^n,\mathcal{Y}\otimes\mathbb{C}^n)$, i.e., for every $n\in\mathbb{Z}_+\setminus\{0\}$ and $\mathbf{Z}\in(\mathbb{C}^{n\times n})^N$ there exist subspaces $\mathcal{D}_{\mathbf{Z}}$ and $\mathcal{D}_{*,\mathbf{Z}}$ in $\widetilde{\mathcal{X}}\otimes\mathbb{C}^n$ such that

$$\widetilde{\mathcal{X}}\otimes\mathbb{C}^n=\mathcal{D}_{\mathbf{Z}}\oplus(\mathcal{X}\otimes\mathbb{C}^n)\oplus\mathcal{D}_{*,\mathbf{Z}},\tag{4.1}$$

$$(\widetilde{\mathbf{A}}\otimes\mathbf{Z})\mathcal{D}_{\mathbf{Z}}\subset\mathcal{D}_{\mathbf{Z}},\ (\widetilde{\mathbf{C}}\otimes\mathbf{Z})\mathcal{D}_{\mathbf{Z}}=\{0\},\ (\widetilde{\mathbf{A}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}\subset\mathcal{D}_{*,\mathbf{Z}},\ (\widetilde{\mathbf{B}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}=\{0\},\tag{4.2}$$

$$\begin{aligned}\mathbf{A}\otimes\mathbf{Z}&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})|(\mathcal{X}\otimes\mathbb{C}^n),\quad &\mathbf{B}\otimes\mathbf{Z}&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{B}}\otimes\mathbf{Z}),\\ \mathbf{C}\otimes\mathbf{Z}&=(\widetilde{\mathbf{C}}\otimes\mathbf{Z})|(\mathcal{X}\otimes\mathbb{C}^n), &\mathbf{D}\otimes\mathbf{Z}&=\widetilde{\mathbf{D}}\otimes\mathbf{Z}.\end{aligned}\tag{4.3}$$

Proposition 4.1. *The system* $\widetilde{\Sigma}^{\rm KV,nc} = (N; \widetilde{U}_z^{\rm KV,nc}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm KV,nc} = (N; U_z^{\rm KV,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}} \supset \mathcal{X}$, $\widetilde{\mathbf{D}} = \mathbf{D}$, *and for all* $j \in \mathbb{Z}_+$ *the following equalities of non-commutative polynomials hold:*

$$\begin{aligned} (z\mathbf{A})^j &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & (z\mathbf{A})^j z\mathbf{B} &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j z\widetilde{\mathbf{B}}, \\ z\mathbf{C}(z\mathbf{A})^j &= z\widetilde{\mathbf{C}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & z\mathbf{C}(z\mathbf{A})^j z\mathbf{B} &= z\widetilde{\mathbf{C}}(z\widetilde{\mathbf{A}})^j z\widetilde{\mathbf{B}}. \end{aligned} \tag{4.4}$$

Proof. From the definition of dilation above and Lemma 1.1 it follows that the system $\widetilde{\Sigma}^{\rm KV,nc}$ is a dilation of the system $\Sigma^{\rm KV,nc}$ if and only if $\widetilde{\mathbf{D}} = \mathbf{D}$ and for every $n \in \mathbb{Z}_+ \setminus \{0\}$, $\mathbf{Z} \in (\mathbb{C}^{n\times n})^N$ and $j \in \mathbb{Z}_+$,

$$\begin{aligned} (\mathbf{A}\otimes\mathbf{Z})^j &= (P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j|(\mathcal{X}\otimes\mathbb{C}^n), \\ (\mathbf{A}\otimes\mathbf{Z})^j(\mathbf{B}\otimes\mathbf{Z}) &= (P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j(\widetilde{\mathbf{B}}\otimes\mathbf{Z}), \\ (\mathbf{C}\otimes\mathbf{Z})(\mathbf{A}\otimes\mathbf{Z})^j &= (\widetilde{\mathbf{C}}\otimes\mathbf{Z})(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j|(\mathcal{X}\otimes\mathbb{C}^n), \\ (\mathbf{C}\otimes\mathbf{Z})(\mathbf{A}\otimes\mathbf{Z})^j(\mathbf{B}\otimes\mathbf{Z}) &= (\widetilde{\mathbf{C}}\otimes\mathbf{Z})(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j(\widetilde{\mathbf{B}}\otimes\mathbf{Z}). \end{aligned}$$

The latter is equivalent to equalities (4.4) due to the non-existence of polynomial identities valid for all matrix rings $\mathbb{C}^{n\times n}$, $n = 1, 2, \ldots$ (see e.g. [25, pp. 22–23]). □

As a corollary of the last of equalities (4.4) in Proposition 4.1 we obtain the following statement.

Proposition 4.2. *Transfer functions* $T_{\Sigma^{\rm KV,nc}}$ *and* $T_{\widetilde{\Sigma}^{\rm KV,nc}}$ *of a system* $\Sigma^{\rm KV,nc} = (N; U_z^{\rm KV,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *and of its dilation* $\widetilde{\Sigma}^{\rm KV,nc} = (N; \widetilde{U}_z^{\rm KV,nc}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *coincide.*

Recall the notation

$$\mathbf{A}^w := A_{i_1}\cdots A_{i_m} \tag{4.5}$$

for $w = g_{i_1}\cdots g_{i_m} \in \mathcal{F}_N \setminus \{\emptyset\}$ and $\mathbf{A} = (A_1, \ldots, A_N) \in \mathcal{L}(\mathcal{X})^N$, and

$$\mathbf{A}^\emptyset = I_{\mathcal{X}}. \tag{4.6}$$

We also introduce the notations for a non-commutative KV system $\Sigma^{\rm KV,nc} = (N; U_z^{\rm KV,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$:

$$(\mathbf{A}\sharp\mathbf{B})^w = A_{i_1}\cdots A_{i_{m-1}}B_{i_m}, \quad w = g_{i_1}\cdots g_{i_m} \in \mathcal{F}_N \setminus \{\emptyset\}, \tag{4.7}$$

$$(\mathbf{C}\flat\mathbf{A})^w = C_{i_1}A_{i_2}\cdots A_{i_m}, \quad w = g_{i_1}\cdots g_{i_m} \in \mathcal{F}_N \setminus \{\emptyset\}, \tag{4.8}$$

$$(\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^w = C_{i_1}A_{i_2}\cdots A_{i_{m-1}}B_{i_m}, \quad w = g_{i_1}\cdots g_{i_m} \in \mathcal{F}_N \setminus \{\emptyset, g_1, \ldots, g_N\}, \tag{4.9}$$

where, in particular,

$$\begin{aligned} (\mathbf{A}\sharp\mathbf{B})^{g_k} &= B_k, \quad k = 1, \ldots, N, \\ (\mathbf{C}\flat\mathbf{A})^{g_k} &= C_k, \quad k = 1, \ldots, N, \\ (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^{g_k g_j} &= C_k B_j, \quad k, j = 1, \ldots, N. \end{aligned}$$

Then, rewriting the equalities (4.4) in Proposition 4.1 as the equalities for the coefficients of non-commutative polynomials, we obtain the following non-commutative analogue of Lemma 1.1.

Proposition 4.3. *The system* $\widetilde{\Sigma}^{\rm KV,nc} = (N; \widetilde{U}_z^{\rm KV,nc}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm KV,nc} = (N; U_z^{\rm KV,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}} \supset \mathcal{X}$, $\widetilde{\mathbf{D}} = \mathbf{D}$, *and the* *following equalities hold:*

$$\begin{aligned} \mathbf{A}^w &= P_{\mathcal{X}}\widetilde{\mathbf{A}}^w|\mathcal{X}, \quad w\in\mathcal{F}_N, & (4.10)\\ (\mathbf{A}\sharp\mathbf{B})^w &= P_{\mathcal{X}}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^w, \quad w\in\mathcal{F}_N\setminus\{\emptyset\}, & (4.11)\\ (\mathbf{C}\flat\mathbf{A})^w &= (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}})^w|\mathcal{X}, \quad w\in\mathcal{F}_N\setminus\{\emptyset\}, & (4.12)\\ (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^w &= (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^w, \quad w\in\mathcal{F}_N\setminus\{\emptyset,g_1,\dots,g_N\}. & (4.13) \end{aligned}$$

Let the non-commutative KV system $\widetilde{\Sigma}^{\rm KV,nc}=(N;\widetilde{U}_z^{\rm KV,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ be a dilation of the non-commutative KV system $\Sigma^{\rm KV,nc}=(N;U_z^{\rm KV,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$. We will call such a dilation *uniform* if the subspaces $\mathcal{D}_{\mathbf{Z}}$ and $\mathcal{D}_{*,\mathbf{Z}}$ in $\widetilde{\mathcal{X}}\otimes\mathbb{C}^n$ from the equalities (4.1) and (4.2) are independent of $\mathbf{Z}$ and have the form $\mathcal{D}_{\mathbf{Z}}=\mathcal{D}\otimes\mathbb{C}^n,\ \mathcal{D}_{*,\mathbf{Z}}=\mathcal{D}_*\otimes\mathbb{C}^n$. Thus, $\widetilde{\Sigma}^{\rm KV,nc}$ is called a *uniform dilation* of $\Sigma^{\rm KV,nc}$ if there exist subspaces $\mathcal{D}$ and $\mathcal{D}_*$ in $\widetilde{\mathcal{X}}$ such that

$$\widetilde{\mathcal{X}}=\mathcal{D}\oplus\mathcal{X}\oplus\mathcal{D}_*, \tag{4.14}$$

$$\widetilde{A_k}\mathcal{D}\subset\mathcal{D},\quad \widetilde{C_k}\mathcal{D}=\{0\},\quad \widetilde{A_k}^*\mathcal{D}_*\subset\mathcal{D}_*,\quad \widetilde{B_k}^*\mathcal{D}_*=\{0\},\quad k=1,\dots,N, \tag{4.15}$$

$$A_k=P_{\mathcal{X}}\widetilde{A_k}|\mathcal{X},\quad B_k=P_{\mathcal{X}}\widetilde{B_k},\quad C_k=\widetilde{C_k}|\mathcal{X},\quad D_k=\widetilde{D}_k,\quad k=1,\dots,N. \tag{4.16}$$

Proposition 4.4. *The system* $\widetilde{\Sigma}^{\rm KV,nc}=(N;\widetilde{U}_z^{\rm KV,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm KV,nc}=(N;U_z^{\rm KV,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ *if and only if* $\widetilde{\Sigma}^{\rm KV,nc}$ *is a uniform dilation of* $\Sigma^{\rm KV,nc}$.

Proof. Clearly, only the necessity part is non-trivial. Suppose that $\widetilde{\Sigma}^{\rm KV,nc}$ is a dilation of $\Sigma^{\rm KV,nc}$. Then by Proposition 4.3, $\widetilde{\mathcal{X}}\supset\mathcal{X},\widetilde{\mathbf{D}}=\mathbf{D}$, and relations (4.10)–(4.13) hold. In particular, (4.10)–(4.12) imply (4.16). Set

$$\mathcal{D}:=\bigvee_{w\in\mathcal{F}_N,\,k\in\{1,\dots,N\}}\widetilde{\mathbf{A}}^w\left((\widetilde{A_k}-A_k)\mathcal{X}+(\widetilde{B_k}-B_k)\mathcal{U}\right),$$

where "$\bigvee_j\mathcal{L}_j$" denotes the closure of the linear span of sets $\mathcal{L}_j$ in a Hilbert space, and

$$(\widetilde{A_k}-A_k)\mathcal{X}+(\widetilde{B_k}-B_k)\mathcal{U}:=\{\widetilde{A_k}x-A_kx+\widetilde{B_k}u-B_ku:\ x\in\mathcal{X},u\in\mathcal{U}\}\subset\widetilde{\mathcal{X}}.$$

Then $\mathcal{D}\perp\mathcal{X}$. Indeed, for arbitrary $x\in\mathcal{X},u\in\mathcal{U},w\in\mathcal{F}_N$, and $k\in\{1,\dots,N\}$ we have, due to (4.10)–(4.11),

$$\begin{aligned} &P_{\mathcal{X}}\widetilde{\mathbf{A}}^w\left((\widetilde{A_k}-A_k)x+(\widetilde{B_k}-B_k)u\right)\\ &= P_{\mathcal{X}}\widetilde{\mathbf{A}}^{wg_k}x-(P_{\mathcal{X}}\widetilde{\mathbf{A}}^w|\mathcal{X})\cdot A_kx+P_{\mathcal{X}}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^{wg_k}u-(P_{\mathcal{X}}\widetilde{\mathbf{A}}^w|\mathcal{X})\cdot B_ku\\ &= 0. \end{aligned}$$

Set

$$\mathcal{D}_*:=\widetilde{\mathcal{X}}\ominus(\mathcal{D}\oplus\mathcal{X}).$$

Then (4.14) holds. From the definition of $\mathcal{D}$ we obtain that $\widetilde{A_j}\mathcal{D}\subset\mathcal{D},\ j=1,\dots,N$. Further, for arbitrary $x\in\mathcal{X},u\in\mathcal{U},w\in\mathcal{F}_N$, and $k,j\in\{1,\dots,N\}$ we have, due to

(4.12)–(4.13),

$$\begin{aligned} & \widetilde{C_j}\widetilde{\mathbf{A}}^w\Big((\widetilde{A_k}-A_k)x+(\widetilde{B_k}-B_k)u\Big) \\ = \ & (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}})^{g_jwg_k}x-\Big((\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}})^{g_jw}|\mathcal{X}\Big)\cdot A_kx \\ + \ & (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^{g_jwg_k}u-\Big((\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}})^{g_jw}|\mathcal{X}\Big)\cdot B_ku \\ = \ & 0. \end{aligned}$$

From here we obtain $\widetilde{C_j}\mathcal{D}=\{0\},\ j=1,\dots,N$. For arbitrary $x\in\mathcal{X}$ and $j\in\{1,\dots,N\}$ we have

$$\widetilde{A_j}x=(\widetilde{A_j}x-A_jx)+A_jx\in\operatorname{clos}\{(\widetilde{A_j}-A_j)\mathcal{X}\}\oplus\mathcal{X}\subset\mathcal{D}\oplus\mathcal{X}.$$

It was shown above that $\widetilde{A_j}\mathcal{D}\subset\mathcal{D}$. Hence $\widetilde{A_j}(\mathcal{D}\oplus\mathcal{X})\subset\mathcal{D}\oplus\mathcal{X}$. From here we get

$$\widetilde{A_j}^*\mathcal{D}_*=\widetilde{A_j}^*(\mathcal{D}\oplus\mathcal{X})^\perp\subset(\mathcal{D}\oplus\mathcal{X})^\perp=\mathcal{D}_*.$$

For arbitrary $u\in\mathcal{U}$ and $j\in\{1,\dots,N\}$ we have

$$\widetilde{B_j}u=(\widetilde{B_j}u-B_ju)+B_ju\in\operatorname{clos}\{(\widetilde{B_j}-B_j)\mathcal{U}\}\oplus\mathcal{X}\subset\mathcal{D}\oplus\mathcal{X}=(\mathcal{D}_*)^\perp.$$

From here we get $\widetilde{B_j}^*\mathcal{D}_*=\{0\},\ j=1,\dots,N$. Thus, relations in (4.15) hold true. Finally, we have obtained that the non-commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ is a uniform dilation of the non-commutative KV system $\Sigma^{\mathrm{KV,nc}}$. □

Theorem 4.5. *Every dissipative non-commutative KV system $\Sigma^{\mathrm{KV,nc}}$ has a (uniform) conservative dilation.*

Proof. Let $\Sigma^{\mathrm{KV,nc}}=(N;U_z^{\mathrm{KV,nc}};\mathcal{X},\mathcal{U},\mathcal{Y})$ be a dissipative non-commutative KV system. Then, as was shown in the proof of Proposition 3.1, for an arbitrary $\mathbf{T}\in\mathcal{D}^N_{\mathrm{matr}}$ one has $\|\mathbf{U}\otimes\mathbf{T}\|<1$, which means that $U_z^{\mathrm{KV,nc}}=z\mathbf{U}\in\mathcal{SA}_N^{nc,0}(\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$. By Proposition 3.1 there exists a conservative non-commutative KV system $\dot{\Sigma}^{\mathrm{KV,nc}}=(N;\dot{U}_z^{\mathrm{KV,nc}};\dot{\mathcal{X}},\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$ such that $T_{\dot{\Sigma}^{\mathrm{KV,nc}}}(z)=U_z^{\mathrm{KV,nc}}$, i.e.,

$$z\dot{\mathbf{D}}+z\dot{\mathbf{C}}(I_{\dot{\mathcal{X}}}-z\dot{\mathbf{A}})^{-1}z\dot{\mathbf{B}}=z\mathbf{U}.$$

Then from the uniqueness of the expansion of formal power series in homogeneous polynomials we get $z\dot{\mathbf{D}}=z\mathbf{U}$ (which means that $\dot{\mathbf{D}}=\mathbf{U}$) and for all $j\in\mathbb{Z}_+$:

$$z\dot{\mathbf{C}}(z\dot{\mathbf{A}})^jz\dot{\mathbf{B}}=0. \tag{4.17}$$

The conservativity of the system $\dot{\Sigma}^{\mathrm{KV,nc}}$ means that the linear polynomial

$$\begin{aligned} \dot{U}_z^{\mathrm{KV,nc}} &= z\dot{\mathbf{U}}=\begin{bmatrix} z\dot{\mathbf{A}} & z\dot{\mathbf{B}} \\ z\dot{\mathbf{C}} & z\dot{\mathbf{D}}\end{bmatrix}=\begin{bmatrix} z\dot{\mathbf{A}} & z\dot{\mathbf{B}} \\ z\dot{\mathbf{C}} & z\mathbf{U}\end{bmatrix} \\ &\in \mathcal{L}(\dot{\mathcal{X}}\oplus(\mathcal{X}\oplus\mathcal{U}),\dot{\mathcal{X}}\oplus(\mathcal{X}\oplus\mathcal{Y}))\langle z_1,\dots,z_N\rangle \end{aligned}$$

(here we denote by $\mathcal{L}\langle z_1,\dots,z_N\rangle$ the linear space of polynomials in non-commuting indeterminates $z_1,\dots,z_N$ with the coefficients in a linear space $\mathcal{L}$) satisfies the condition that $\dot{U}_{\mathbf{V}}^{\mathrm{KV,nc}}$ is a unitary operator for every $\mathbf{V}\in\mathcal{U}^N$. This linear polynomial allows another partition:

$$\begin{aligned} \dot{U}_z^{\mathrm{KV,nc}} &= \widetilde{U}_z^{\mathrm{KV,nc}}=z\widetilde{\mathbf{U}}=\begin{bmatrix} z\widetilde{\mathbf{A}} & z\widetilde{\mathbf{B}} \\ z\widetilde{\mathbf{C}} & z\widetilde{\mathbf{D}}\end{bmatrix} \\ &\in \mathcal{L}((\dot{\mathcal{X}}\oplus\mathcal{X})\oplus\mathcal{U},(\dot{\mathcal{X}}\oplus\mathcal{X})\oplus\mathcal{Y})\langle z_1,\dots,z_N\rangle, \end{aligned}$$

where

$$\begin{array}{ll} z\widetilde{\mathbf{A}}=\begin{bmatrix} z\dot{\mathbf{A}} & z\dot{\mathbf{B}}|\mathcal{X} \\ P_{\mathcal{X}}(z\dot{\mathbf{C}}) & z\mathbf{A} \end{bmatrix}, & z\widetilde{\mathbf{B}}=\begin{bmatrix} (z\dot{\mathbf{B}})|\mathcal{U} \\ z\mathbf{B} \end{bmatrix}, \\ z\widetilde{\mathbf{C}}=\begin{bmatrix} P_{\mathcal{Y}}(z\dot{\mathbf{C}}) & z\mathbf{C} \end{bmatrix}, & z\widetilde{\mathbf{D}}=z\mathbf{D}. \end{array} \tag{4.18}$$

It is clear that one can associate to this partition a conservative non-commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,nc}}=(N;\widetilde{U}_z^{\mathrm{KV,nc}};\dot{\mathcal{X}}\oplus\mathcal{X},\mathcal{U},\mathcal{Y})$. Let us show that $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ is a dilation of $\Sigma^{\mathrm{KV,nc}}$. To this end, according to Proposition 4.1, it is sufficient to verify the equalities in (4.4). According to (4.18), $z\mathbf{A}=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})|\mathcal{X}$, i.e. for $j=1$ the first equality in (4.4) holds (for $j=0$ it holds trivially). Let us apply induction on j. Suppose that $(z\mathbf{A})^j=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}$ for $j=k\in\mathbb{Z}_+\setminus\{0\}$. Then by (4.18) and (4.17) we have

$$\begin{aligned} (z\mathbf{A})^{k+1} &= (z\mathbf{A})(z\mathbf{A})^k=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^k|\mathcal{X} \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})(I_{\dot{\mathcal{X}}\oplus\mathcal{X}}-P_{\dot{\mathcal{X}}})(z\widetilde{\mathbf{A}})^k|\mathcal{X} \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}-P_{\mathcal{X}}(z\dot{\mathbf{C}})P_{\dot{\mathcal{X}}}(z\widetilde{\mathbf{A}})^k|\mathcal{X} \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}-P_{\mathcal{X}}(z\dot{\mathbf{C}})\begin{bmatrix} z\dot{\mathbf{A}} & (z\dot{\mathbf{B}})|\mathcal{X} \end{bmatrix}(z\widetilde{\mathbf{A}})^{k-1}|\mathcal{X} \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}-P_{\mathcal{X}}(z\dot{\mathbf{C}})(z\dot{\mathbf{A}})P_{\dot{\mathcal{X}}}(z\widetilde{\mathbf{A}})^{k-1}|\mathcal{X}=\dots \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}-P_{\mathcal{X}}(z\dot{\mathbf{C}})(z\dot{\mathbf{A}})^{k-1}P_{\dot{\mathcal{X}}}(z\widetilde{\mathbf{A}})|\mathcal{X} \\ &= P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}-P_{\mathcal{X}}(z\dot{\mathbf{C}})(z\dot{\mathbf{A}})^{k-1}(z\dot{\mathbf{B}})|\mathcal{X}=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^{k+1}|\mathcal{X}. \end{aligned}$$

Thus the first equality in (4.4) is valid for all $j\in\mathbb{Z}_+$. Other equalities in (4.4) are proved analogously. Therefore, $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ is a dilation of $\Sigma^{\mathrm{KV,nc}}$. According to Proposition 4.4, $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ is a uniform dilation of $\Sigma^{\mathrm{KV,nc}}$. □

4.2. **The commutative setting.** Let us recall (see [20]) that the commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,c}}=(N;\widetilde{U}_z^{\mathrm{KV,c}};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is said to be a *dilation of the commutative KV system* $\Sigma^{\mathrm{KV,c}}=(N;U_z^{\mathrm{KV,c}};\mathcal{X},\mathcal{U},\mathcal{Y})$ if for every fixed $z\in\mathbb{C}^N$ the 1D system $\widetilde{\Sigma}_z^{\mathrm{KV,c}}:=(1;z\widetilde{\mathbf{A}},z\widetilde{\mathbf{B}},z\widetilde{\mathbf{C}},z\widetilde{\mathbf{D}};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a dilation of the 1D system $\Sigma_z^{\mathrm{KV,c}}:=(1;z\mathbf{A},z\mathbf{B},z\mathbf{C},z\mathbf{D};\mathcal{X},\mathcal{U},\mathcal{Y})$, i.e. for each $z\in\mathbb{C}^N$ there exist subspaces $\mathcal{D}_z$ and $\mathcal{D}_{*,z}$ in $\widetilde{\mathcal{X}}$ such that

$$\widetilde{\mathcal{X}}=\mathcal{D}_z\oplus\mathcal{X}\oplus\mathcal{D}_{*,z}, \tag{4.19}$$

$$z\widetilde{\mathbf{A}}\mathcal{D}_z\subset\mathcal{D}_z,\ z\widetilde{\mathbf{C}}\mathcal{D}_z=\{0\},\ (z\widetilde{\mathbf{A}})^*\mathcal{D}_{*,z}\subset\mathcal{D}_{*,z},\ (z\widetilde{\mathbf{B}})^*\mathcal{D}_{*,z}=\{0\}, \tag{4.20}$$

$$z\mathbf{A}=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})|\mathcal{X},\ z\mathbf{B}=P_{\mathcal{X}}(z\widetilde{\mathbf{B}}),\ z\mathbf{C}=(z\widetilde{\mathbf{C}})|\mathcal{X},\ z\mathbf{D}=z\widetilde{\mathbf{D}}. \tag{4.21}$$

From Lemma 1.1 we obtain the following equivalent reformulation of this definition (see Proposition 3.2 and Remark 3.6 in [20]).

Proposition 4.6. *The system* $\widetilde{\Sigma}^{\mathrm{KV,c}}=(N;\widetilde{U}_z^{\mathrm{KV,c}};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ *is a dilation of the system* $\Sigma^{\mathrm{KV,c}}=(N;U_z^{\mathrm{KV,c}};\mathcal{X},\mathcal{U},\mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}}\supset\mathcal{X}$, $\widetilde{\mathbf{D}}=\mathbf{D}$, *and for all* $z\in\mathbb{C}^N$ *and* $j\in\mathbb{Z}_+$ *the following equalities hold:*

$$\begin{array}{rr} (z\mathbf{A})^j=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & (z\mathbf{A})^jz\mathbf{B}=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^jz\widetilde{\mathbf{B}}, \\ z\mathbf{C}(z\mathbf{A})^j=z\widetilde{\mathbf{C}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & z\mathbf{C}(z\mathbf{A})^jz\mathbf{B}=z\widetilde{\mathbf{C}}(z\widetilde{\mathbf{A}})^jz\widetilde{\mathbf{B}}. \end{array} \tag{4.22}$$

Note that the equalities (4.22) are equivalent to the same equalities for the commuting indeterminates $z=(z_1,\dots,z_N)$ in the place of N-tuples $z=(z_1,\dots,z_N)$ of complex numbers, which are understood in this case as equalities of commutative (formal) polynomials, or equivalently, the corresponding equalities for their coefficients. For convenience of writing those relations for coefficients, let us recall the following notations for *symmetrized multipowers of operator tuples* introduced in [19] (we are using here a bit different, however equivalent, description of these notations). Define the *abelianization map* $\gamma : \mathcal{F}_N \to \mathbb{Z}_+^N$ by $w = g_{i_1} \cdots g_{i_m} \mapsto s = (s_1, \dots, s_N)$ where $s_k \in \mathbb{Z}_+$ is the number of times that the letter g_k appears in the word w, and $\emptyset \mapsto 0$. For $s = (s_1, \dots, s_N) \in \mathbb{Z}_+^N$ denote by c_s the cardinality of the set $\gamma^{-1}(s)$. Clearly,

$$c_s := \frac{(s_1 + \cdots + s_N)!}{s_1! \cdots s_N!}.$$

We set

$$\mathbf{A}^0 := I_{\mathcal{X}}, \tag{4.23}$$

$$\mathbf{A}^s := c_s^{-1} \sum_{w \in \gamma^{-1}(s)} \mathbf{A}^w, \quad s \in \mathbb{Z}_+^N \setminus \{0\}, \tag{4.24}$$

$$(\mathbf{A}\sharp\mathbf{B})^s := c_s^{-1} \sum_{w \in \gamma^{-1}(s)} (\mathbf{A}\sharp\mathbf{B})^w, \quad s \in \mathbb{Z}_+^N \setminus \{0\}, \tag{4.25}$$

$$(\mathbf{C}\flat\mathbf{A})^s := c_s^{-1} \sum_{w \in \gamma^{-1}(s)} (\mathbf{C}\flat\mathbf{A})^w, \quad s \in \mathbb{Z}_+^N \setminus \{0\}, \tag{4.26}$$

$$(\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^s := c_s^{-1} \sum_{w \in \gamma^{-1}(s)} (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^s, \tag{4.27}$$

$$s \in \mathbb{Z}_+^N \setminus \{0, e_1, \dots, e_N\}, \text{where } e_1 = (1, 0, \dots, 0), \text{etc.},$$

where we make use of (4.5), (4.7)–(4.9) on the right-hand sides of these formulas.

Remark 4.7. In case of the commutative N-tuple $\mathbf{A}$ we have

$$\mathbf{A}^s = \prod_{k=1}^N {A_k}^{s_k}$$

i.e., a usual multipower.

Thus, the following commutative N-D version of the Sarason lemma is true (see [20, Proposition 3.4]).

Proposition 4.8. *The system $\widetilde{\Sigma}^{\mathrm{KV,c}} = (N; \widetilde{U}_z^{\mathrm{KV,c}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a dilation of the system $\Sigma^{\mathrm{KV,c}} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ if and only if $\widetilde{\mathcal{X}} \supset \mathcal{X}$, $\widetilde{\mathbf{D}} = \mathbf{D}$, and the following equalities hold:*

$$\begin{array}{rc} \forall s \in \mathbb{Z}_+^N & \mathbf{A}^s = P_{\mathcal{X}} \widetilde{\mathbf{A}}^s | \mathcal{X}, \\ \forall s \in \mathbb{Z}_+^N \setminus \{0\} & (\mathbf{A}\sharp\mathbf{B})^s = P_{\mathcal{X}} (\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^s, \\ \forall s \in \mathbb{Z}_+^N \setminus \{0\} & (\mathbf{C}\flat\mathbf{A})^s = (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}})^s | \mathcal{X}, \\ \forall s \in \mathbb{Z}_+^N \setminus \{0, e_1, \dots, e_N\} & (\mathbf{C}\flat\mathbf{A}\sharp\mathbf{B})^s = (\widetilde{\mathbf{C}}\flat\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^s. \end{array} \tag{4.28}$$

As a corollary of the last of equalities (4.22) in Proposition 4.6 we obtain the following statement.

Proposition 4.9. *The transfer functions $T_{\Sigma^{\mathrm{KV,c}}}$ and $T_{\widetilde{\Sigma}^{\mathrm{KV,c}}}$ of a system $\Sigma^{\mathrm{KV,c}} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and of its dilation $\widetilde{\Sigma}^{\mathrm{KV,c}} = (N; \widetilde{U}_z^{\mathrm{KV,c}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, respectively, coincide.*

The following criterion for the existence of a conservative dilation of a dissipative commutative KV system has been obtained in [20, Theorem 4.2].

Theorem 4.10. *A dissipative system $\Sigma^{\mathrm{KV,c}} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ has a conservative dilation if and only if the linear polynomial*

$$U_z^{\mathrm{KV,c}} = z\mathbf{U} = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ z\mathbf{C} & z\mathbf{D} \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})[z_1, \dots, z_N]$$

(here we use notation $\mathcal{L}[z_1, \dots, z_N]$ for the linear space of polynomials in commuting indeterminates $z_1, \dots, z_N$ with the coefficients in a linear space $\mathcal{L}$) belongs to the class $\mathcal{SA}_N^0(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$.

Making use of this criterion and of the result of [23] on non-validity of the von Neumann inequality for linear matrix-valued functions of more than two variables, it has been shown in [20] that for $N \geq 3$ not all dissipative commutative KV systems have conservative dilations.

Let the system $\widetilde{\Sigma}^{\mathrm{KV,c}} = (N; \widetilde{U}_z^{\mathrm{KV,c}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ be a dilation of the system $\Sigma^{\mathrm{KV,c}} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. We will say that such a dilation is *uniform* if the subspaces $\mathcal{D}_z$ are independent of $z \in \mathbb{C}^N$, i.e., $\mathcal{D}_z = \mathcal{D},\ z \in \mathbb{C}^N$, or equivalently, subspaces $\mathcal{D}_{*,z}$ are independent of $z \in \mathbb{C}^N$, i.e., $\mathcal{D}_{*,z} = \mathcal{D},\ z \in \mathbb{C}^N$. The following theorem is an improvement of Theorem 4.10.

Theorem 4.11. *A dissipative system $\Sigma^{\mathrm{KV,c}} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ has a uniform conservative dilation if and only if the linear polynomial $U_z^{\mathrm{KV,c}}$ belongs to the class $\mathcal{SA}_N^0(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$.*

Proof. The necessity part follows from Theorem 4.10. For the proof of the sufficiency part, assume that $U_z^{\mathrm{KV,c}}$ belongs to the class $\mathcal{SA}_N^0(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$. This means that for any N-tuple $\mathbf{T} = (T_1, \dots, T_N)$ of commuting strict contractions on a common Hilbert space one has $\|U_{\mathbf{T}}^{\mathrm{KV,c}}\| \leq 1$. Let $\mathbf{V} = (V_1, \dots, V_N)$ be an N-tuple of (not necessarily commuting) unitary operators on a common Hilbert space, say $\mathcal{H}$. Then for an arbitrary $r:\ 0 < r < 1$, the N-tuple $r\mathbf{V} = (rV_1, \dots, rV_N)$ consists of (not necessarily commuting) strict contractions on $\mathcal{H}$. The operators

$$T_k := \begin{bmatrix} 0 & V_k \\ 0 & 0 \end{bmatrix}, \qquad k = 1, \dots, N,$$

are commuting contractions on $\mathcal{H} \oplus \mathcal{H}$: for $k, j = 1, \dots, N$ one has $T_k T_j = 0$. Therefore for an arbitrary $r:\ 0 < r < 1$, the N-tuple $r\mathbf{T} = (rT_1, \dots, rT_N)$ consists of commuting strict contractions on $\mathcal{H} \oplus \mathcal{H}$. Therefore,

$$\|U_{\mathbf{V}}^{\mathrm{KV,nc}}\| = \lim_{r \to 1} \|U_{r\mathbf{V}}^{\mathrm{KV,nc}}\| = \lim_{r \to 1} \|U_{r\mathbf{T}}^{\mathrm{KV,c}}\| \leq 1,$$

thus the associated non-commutative KV system $\Sigma^{\mathrm{KV,nc}} = (N; U_z^{\mathrm{KV,nc}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is dissipative. According to Theorem 4.5, $\Sigma^{\mathrm{KV,nc}}$ has a uniform conservative dilation $\widetilde{\Sigma}^{\mathrm{KV,nc}} = (N; \widetilde{U}_z^{\mathrm{KV,nc}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, i.e., the system $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ is conservative and (4.14)–(4.16) hold. Then the commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,c}} = (N; \widetilde{U}_z^{\mathrm{KV,c}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, which corresponds to $\widetilde{\Sigma}^{\mathrm{KV,nc}}$ and which is conservative as was noted in Section 3.1, is a uniform dilation of the original commutative KV system $\Sigma^{\mathrm{KV,c}}$. □

Let us make an obvious, however important, remark that the commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,c}} = (N; \widetilde{U}_z^{\mathrm{KV,c}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a uniform dilation of the commutative KV system $\Sigma^{KV,c} = (N; U_z^{\mathrm{KV,c}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ if and only if the associated non-commutative KV system $\widetilde{\Sigma}^{\mathrm{KV,nc}} = (N; \widetilde{U}_z^{\mathrm{KV,nc}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a uniform dilation of the associated non-commutative KV system $\Sigma^{\mathrm{KV,nc}} = (N; U_z^{\mathrm{KV,nc}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The next example shows that in the commutative case not every dilation is uniform (in contrast with the non-commutative case, see Proposition 4.4). Moreover, not every conservative dilation is uniform, which means that Theorem 4.11 is indeed an improvement of Theorem 4.10.

Example 4.12. Let us define systems

$$\Sigma^{\rm KV} := (2;\, 0;\, \mathbb{C}, \{0\}, \{0\}), \qquad \dot{\Sigma}^{\rm KV} := (2;\, \dot{U}_z^{\rm KV} = z\dot{\mathbf{A}};\, \mathbb{C}^3, \{0\}, \{0\}),$$

where

$$z\dot{\mathbf{A}} := \begin{bmatrix} 0 & 0 & z_2/\sqrt{2} \\ 0 & 0 & z_1/\sqrt{2} \\ z_1/\sqrt{2} & -z_2/\sqrt{2} & 0 \end{bmatrix},$$

and the state space of the first system, $\mathcal{X} = \mathbb{C}$, is identified with the subspace $\operatorname{col}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbb{C}$ in $\dot{\mathcal{X}} = \mathbb{C}^3$. Then it is easy to see that for a pair of commuting indeterminates $z = (z_1, z_2)$ one has

$$P_{\mathcal{X}}(z\dot{\mathbf{A}})^j|\mathcal{X} = ((z\dot{\mathbf{A}})^j)_{33} = 0 = z\mathbf{A}, \quad j = 1, 2, \ldots,$$

which means due to Proposition 4.6 that the commutative KV system $\dot{\Sigma}^{\rm KV,c}$ is a dilation of the commutative KV system $\Sigma^{\rm KV,c}$. However, for a pair of non-commuting indeterminates $z = (z_1, z_2)$ one has

$$P_{\mathcal{X}}(z\dot{\mathbf{A}})^2|\mathcal{X} = ((z\dot{\mathbf{A}})^2)_{33} = (z_1z_2 - z_2z_1)/2 \neq 0 = (z\mathbf{A})^2,$$

which means due to Proposition 4.1 that the non-commutative KV system $\dot{\Sigma}^{\rm KV,nc}$ is not a dilation of the non-commutative KV system $\Sigma^{\rm KV,nc}$, and thus by Proposition 4.4 and by the remark preceding this example, the commutative KV system $\dot{\Sigma}^{\rm KV,c}$ is not a uniform dilation of the commutative KV system $\Sigma^{\rm KV,c}$. Now set $\widetilde{\Sigma}^{\rm KV} := (2; \widetilde{U}_z^{\rm KV} = z\widetilde{\mathbf{A}}; \bigoplus_{j=-\infty}^{\infty}\mathbb{C} = \left(\bigoplus_{j=-\infty}^{0}\mathbb{C}\right) \oplus \dot{\mathcal{X}} \oplus \left(\bigoplus_{j=4}^{\infty}\mathbb{C}\right), \{0\}, \{0\})$, where

$$z\widetilde{\mathbf{A}} := \begin{bmatrix} \ddots &&&&&&& \\ & z_1 &&&&&& \\ && z_1/\sqrt{2} & z_2/\sqrt{2} & 0 & 0 && \\ && 0 & 0 & z_2/\sqrt{2} & z_2/\sqrt{2} && \\ && 0 & 0 & z_1/\sqrt{2} & -z_1/\sqrt{2} && \\ && z_1/\sqrt{2} & -z_2/\sqrt{2} & \boxed{0} & 0 && \\ &&&&&& z_1 & \\ &&&&&&& \ddots \end{bmatrix}$$

(here the $(3,3)$-block, i.e., the $\mathcal{L}(\mathcal{X},\mathcal{X})$-block is indicated by a frame). One can easily check that the commutative KV system $\widetilde{\Sigma}^{\rm KV,c}$ is a uniform conservative dilation of the commutative KV system $\dot{\Sigma}^{\rm KV,c}$. Then $\widetilde{\Sigma}^{\rm KV,c}$ is a conservative dilation of $\Sigma^{\rm KV,c}$, since the dilation of a dilation is again a dilation. However, $\widetilde{\Sigma}^{\rm KV,c}$ is not a uniform dilation of $\Sigma^{\rm KV,c}$. Indeed, for a pair of non-commuting indeterminates $z = (z_1, z_2)$ one has

$$P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^2|\mathcal{X} = ((z\widetilde{\mathbf{A}})^2)_{33} = (z_1z_2 - z_2z_1)/2 \neq 0 = (z\mathbf{A})^2,$$

where again we make use of Propositions 4.1 and 4.4, and by the remark preceding this example.

5. Dilations of FM systems

5.1. **The non-commutative setting.** We will say that the non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a *dilation of the non-commutative FM system* $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ if for every $n\in\mathbb{Z}_+\backslash\{0\}$ and $\mathbf{Z}=(Z_1,\ldots,Z_N)\in(\mathbb{C}^{n\times n})^N$ the 1D system $\widetilde{\Sigma}_{\mathbf{Z}}^{\rm FM,nc}:=(1;\widetilde{\mathbf{A}}\otimes\mathbf{Z},\widetilde{\mathbf{B}}\otimes\mathbf{Z},\widetilde{C}\otimes I_n,\widetilde{D}\otimes I_n;\widetilde{\mathcal{X}}\otimes\mathbb{C}^n,\mathcal{U}\otimes\mathbb{C}^n,\mathcal{Y}\otimes\mathbb{C}^n)$ is a dilation of the 1D system $\Sigma_{\mathbf{Z}}^{\rm FM,nc}:=(1;\mathbf{A}\otimes\mathbf{Z},\mathbf{B}\otimes\mathbf{Z},C\otimes I_n,D\otimes I_n;\mathcal{X}\otimes\mathbb{C}^n,\mathcal{U}\otimes\mathbb{C}^n,\mathcal{Y}\otimes\mathbb{C}^n)$, i.e., for every $n\in\mathbb{Z}_+\setminus\{0\}$ and $\mathbf{Z}\in(\mathbb{C}^{n\times n})^N$ there exist subspaces $\mathcal{D}_{\mathbf{Z}}$ and $\mathcal{D}_{*,\mathbf{Z}}$ in $\widetilde{\mathcal{X}}\otimes\mathbb{C}^n$ such that

$$\widetilde{\mathcal{X}}\otimes\mathbb{C}^n=\mathcal{D}_{\mathbf{Z}}\oplus(\mathcal{X}\otimes\mathbb{C}^n)\oplus\mathcal{D}_{*,\mathbf{Z}},\tag{5.1}$$

$$(\widetilde{\mathbf{A}}\otimes\mathbf{Z})\mathcal{D}_{\mathbf{Z}}\subset\mathcal{D}_{\mathbf{Z}},\ (\widetilde{C}\otimes I_n)\mathcal{D}_{\mathbf{Z}}=\{0\},\ (\widetilde{\mathbf{A}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}\subset\mathcal{D}_{*,\mathbf{Z}},\ (\widetilde{\mathbf{B}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}=\{0\},\tag{5.2}$$

$$\begin{aligned}\mathbf{A}\otimes\mathbf{Z}&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})|(\mathcal{X}\otimes\mathbb{C}^n),\quad &\mathbf{B}\otimes\mathbf{Z}&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{B}}\otimes\mathbf{Z}),\\ C\otimes I_n&=(\widetilde{C}\otimes I_n)|(\mathcal{X}\otimes\mathbb{C}^n), &D\otimes I_n&=\widetilde{D}\otimes I_n.\end{aligned}\tag{5.3}$$

Proposition 5.1. *The system $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a dilation of the system $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ if and only if $\widetilde{\mathcal{X}}\supset\mathcal{X}$, $\widetilde{D}=D$, and for all $j\in\mathbb{Z}_+$ the following equalities of non-commutative polynomials hold:*

$$\begin{aligned}(z\mathbf{A})^j&=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j|\mathcal{X},\quad &(z\mathbf{A})^jz\mathbf{B}&=P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^jz\widetilde{\mathbf{B}},\\ C(z\mathbf{A})^j&=\widetilde{C}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, &C(z\mathbf{A})^jz\mathbf{B}&=\widetilde{C}(z\widetilde{\mathbf{A}})^jz\widetilde{\mathbf{B}}.\end{aligned}\tag{5.4}$$

Proof. From the definition of dilation above and Lemma 1.1 it follows that the system $\widetilde{\Sigma}^{\rm FM,nc}$ is a dilation of the system $\Sigma^{\rm FM,nc}$ if and only if for every $n\in\mathbb{Z}_+\backslash\{0\}$, $\mathbf{Z}\in(\mathbb{C}^{n\times n})^N$ and $j\in\mathbb{Z}_+$,

$$\begin{aligned}(\mathbf{A}\otimes\mathbf{Z})^j&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j|(\mathcal{X}\otimes\mathbb{C}^n),\\ (\mathbf{A}\otimes\mathbf{Z})^j(\mathbf{B}\otimes\mathbf{Z})&=(P_{\mathcal{X}}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j(\widetilde{\mathbf{B}}\otimes\mathbf{Z}),\\ (C\otimes I_n)(\mathbf{A}\otimes\mathbf{Z})^j&=(\widetilde{C}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j|(\mathcal{X}\otimes\mathbb{C}^n),\\ (C\otimes I_n)(\mathbf{A}\otimes\mathbf{Z})^j(\mathbf{B}\otimes\mathbf{Z})&=(\widetilde{C}\otimes I_n)(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^j(\widetilde{\mathbf{B}}\otimes\mathbf{Z}).\end{aligned}$$

The latter is equivalent to equalities (5.4) due to the fact that there are no polynomial identities valid for matrices of all sizes, see [25, pp. 22–23]. ☐

As a corollary of the last of equalities (5.4) in Proposition 5.1 we obtain the following statement.

Proposition 5.2. *The transfer functions $T_{\Sigma^{\rm FM,nc}}$ and $T_{\widetilde{\Sigma}^{\rm FM,nc}}$ of a system $\Sigma^{\rm FM,nc}$ $=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ and of its dilation $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ coincide.*

Rewriting the equalities (5.4) in Proposition 5.1 as equalities for the coefficients of polynomials, we obtain the following non-commutative analogue of Lemma 1.1.

Proposition 5.3. *The system $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a dilation of the system $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ if and only if $\widetilde{\mathcal{X}}\supset\mathcal{X}$, $\widetilde{D}=D$, and the* *following equalities hold:*

$$\begin{aligned}\mathbf{A}^w &= P_{\mathcal{X}}\widetilde{\mathbf{A}}^w|\mathcal{X}, \quad w\in\mathcal{F}_N, &(5.5)\\ (\mathbf{A}\sharp\mathbf{B})^w &= P_{\mathcal{X}}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^w, \quad w\in\mathcal{F}_N\setminus\{\emptyset\}, &(5.6)\\ C\mathbf{A}^w &= \widetilde{C}\widetilde{\mathbf{A}}^w|\mathcal{X}, \quad w\in\mathcal{F}_N, &(5.7)\\ C(\mathbf{A}\sharp\mathbf{B})^w &= \widetilde{C}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^w, \quad w\in\mathcal{F}_N\setminus\{\emptyset\}. &(5.8)\end{aligned}$$

Let the non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ be a dilation of the non-commutative FM system $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$. We will call such a dilation *uniform* if the subspaces $\mathcal{D}_{\mathbf{Z}}$ and $\mathcal{D}_{*,\mathbf{Z}}$ in $\widetilde{\mathcal{X}}\otimes\mathbb{C}^n$ from the equalities (5.1) and (5.2) are independent of $\mathbf{Z}$ and have the form $\mathcal{D}_{\mathbf{Z}}=\mathcal{D}\otimes\mathbb{C}^n$, $\mathcal{D}_{*,\mathbf{Z}}=\mathcal{D}_*\otimes\mathbb{C}^n$. Thus, $\widetilde{\Sigma}^{\rm FM,nc}$ is called a *uniform dilation* of $\Sigma^{\rm FM,nc}$ if there exist subspaces $\mathcal{D}$ and $\mathcal{D}_*$ in $\widetilde{\mathcal{X}}$ such that

$$\widetilde{\mathcal{X}}=\mathcal{D}\oplus\mathcal{X}\oplus\mathcal{D}_*, \tag{5.9}$$

$$\widetilde{A_k}\mathcal{D}\subset\mathcal{D},\quad \widetilde{C}\mathcal{D}=\{0\},\quad \widetilde{A_k}^*\mathcal{D}_*\subset\mathcal{D}_*,\quad \widetilde{B_k}^*\mathcal{D}_*=\{0\},\quad k=1,\dots,N, \tag{5.10}$$

$$A_k=P_{\mathcal{X}}\widetilde{A_k}|\mathcal{X},\quad B_k=P_{\mathcal{X}}\widetilde{B_k},\quad C=\widetilde{C}|\mathcal{X},\quad D=\widetilde{D},\quad k=1,\dots,N. \tag{5.11}$$

Proposition 5.4. *The system* $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ *if and only if* $\widetilde{\Sigma}^{\rm FM,nc}$ *is a uniform dilation of* $\Sigma^{\rm FM,nc}$.

Proof. Clearly, only the necessity part is non-trivial. Suppose that $\widetilde{\Sigma}^{\rm FM,nc}$ is a dilation of $\Sigma^{\rm FM,nc}$. Then by Proposition 5.3 $\widetilde{\mathcal{X}}\supset\mathcal{X}$ and relations (5.5)–(5.8) hold. In particular, (5.5)–(5.7) imply (5.11). Set

$$\mathcal{D}:=\bigvee_{w\in\mathcal{F}_N,\,k\in\{1,\dots,N\}}\widetilde{\mathbf{A}}^w\left((\widetilde{A_k}-A_k)\mathcal{X}+(\widetilde{B_k}-B_k)\mathcal{U}\right),$$

Then $\mathcal{D}\perp\mathcal{X}$ (see the proof of Proposition 4.4). Set

$$\mathcal{D}_*:=\widetilde{\mathcal{X}}\ominus(\mathcal{D}\oplus\mathcal{X}).$$

Then (5.9) holds. From the definition of $\mathcal{D}$ we obtain that $\widetilde{A_j}\mathcal{D}\subset\mathcal{D}$, $j=1,\dots,N$. Further, for arbitrary $x\in\mathcal{X},u\in\mathcal{U},w\in\mathcal{F}_N$, and $k,j\in\{1,\dots,N\}$ we have, due to (5.7)–(5.8),

$$\begin{aligned}&\widetilde{C}\widetilde{\mathbf{A}}^w\left((\widetilde{A_k}-A_k)x+(\widetilde{B_k}-B_k)u\right)\\ &= \widetilde{C}\widetilde{\mathbf{A}}^{wg_k}x-\left(\widetilde{C}\widetilde{\mathbf{A}}^w|\mathcal{X}\right)\cdot A_kx\\ &+ \widetilde{C}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^{wg_k}u-\left(\widetilde{C}\widetilde{\mathbf{A}}^w|\mathcal{X}\right)\cdot B_ku\\ &= 0.\end{aligned}$$

From here we obtain $\widetilde{C}\mathcal{D}=\{0\}$. Other relations in (5.10) are obtained in the same way as in the proof of Proposition 4.4. Thus, the non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc}$ is a uniform dilation of the non-commutative FM system $\Sigma^{\rm FM,nc}$. □

Theorem 5.5. *Every* $\mathcal{D}^N$*-dissipative non-commutative FM system* $\Sigma^{\rm FM,nc}$ *has a (uniform)* $\mathcal{D}^N$*-conservative dilation.*

Proof. Let the non-commutative FM system $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$ be $\mathcal{D}^N$-dissipative. Then (see the proof of Proposition 3.4) $U_z^{\rm FM,nc}\in\mathcal{SA}_N^{nc}(\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$. By [10], there exists a conservative non-commutative GR system $\widehat{\Sigma}^{\rm GR,nc}=(N;\widehat{U}_z^{\rm GR,nc};\widehat{\mathcal{X}}=\bigoplus_{k=1}^N\widehat{\mathcal{X}}_k,\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$ such that

$$T_{\widehat{\Sigma}^{\rm GR,nc}}(z)=\hat{D}+\hat{C}(I_{\widehat{\mathcal{X}}}-z\widehat{\mathbf{P}}\hat{A})^{-1}z\widehat{\mathbf{P}}\hat{B}=U_z^{\rm FM,nc},$$

where $\widehat{\mathbf{P}}=(\hat{P}_1,\dots,\hat{P}_N)\in\mathcal{L}(\hat{\mathcal{X}})^N$, with $\hat{P}_k:=P_{\widehat{\mathcal{X}}_k},\ k=1,\dots,N$. Then, by virtue of the uniqueness of homogeneous polynomial expansions of formal power series (this follows, e.g., from the lack of any polynomial identities holding on all square matrices of arbitrary finite size), we get:

$$\hat{D}=\begin{bmatrix}0&0\\C&D\end{bmatrix},\ \hat{C}z\widehat{\mathbf{P}}\hat{B}=\begin{bmatrix}z\mathbf{A}&z\mathbf{B}\\0&0\end{bmatrix},\ \hat{C}(z\widehat{\mathbf{P}}\hat{A})^jz\widehat{\mathbf{P}}\hat{B}=0\ (j=1,\dots).\quad(5.12)$$

The last two relations can be rewritten as

$$\hat{C}\underline{B}_k=\begin{bmatrix}A_k&B_k\\0&0\end{bmatrix}\ (k=1,\dots,N),\quad \hat{C}(\underline{\mathbf{A}}\sharp\underline{\mathbf{B}})^w=0\quad(w\in\mathcal{F}_N\setminus\{\emptyset,g_1,\dots,g_N\}),\quad(5.13)$$

where $\underline{\mathbf{A}}=(\underline{A}_1,\dots,\underline{A}_N)\in\mathcal{L}(\widehat{\mathcal{X}})^N$, with $\underline{A}_k:=\widehat{P}_k\hat{A},\ k=1,\dots,N$, and $\underline{\mathbf{B}}=(\underline{B}_1,\dots,\underline{B}_N)\in\mathcal{L}(\mathcal{X}\oplus\mathcal{U},\widehat{\mathcal{X}})^N$, with $\underline{B}_k:=\widehat{P}_k\hat{B},\ k=1,\dots,N$. The unitary operator

$$\hat{U}=\begin{bmatrix}\hat{A}&\hat{B}\\\hat{C}&\hat{D}\end{bmatrix}\in\mathcal{L}(\widehat{\mathcal{X}}\oplus(\mathcal{X}\oplus\mathcal{U}),\widehat{\mathcal{X}}\oplus(\mathcal{X}\oplus\mathcal{Y}))$$

admits another block partition:

$$\hat{U}=\dot{U}=\begin{bmatrix}\dot{A}&\dot{B}\\\dot{C}&\dot{D}\end{bmatrix}\in\mathcal{L}((\widehat{\mathcal{X}}\oplus\mathcal{X})\oplus\mathcal{U},(\widehat{\mathcal{X}}\oplus\mathcal{X})\oplus\mathcal{Y}),$$

where

$$\begin{aligned}\dot{A}&=\begin{bmatrix}\hat{A}&\hat{B}|\mathcal{X}\\P_{\mathcal{X}}\hat{C}&0\end{bmatrix}\in\mathcal{L}(\widehat{\mathcal{X}}\oplus\mathcal{X}),\quad \dot{B}=\begin{bmatrix}\hat{B}|\mathcal{U}\\0\end{bmatrix}\in\mathcal{L}(\mathcal{U},\widehat{\mathcal{X}}\oplus\mathcal{X}),\\ \dot{C}&=\begin{bmatrix}P_{\mathcal{Y}}\hat{C}&C\end{bmatrix}\in\mathcal{L}(\widehat{\mathcal{X}}\oplus\mathcal{X},\mathcal{Y}),\qquad \dot{D}=D\in\mathcal{L}(\mathcal{U},\mathcal{Y}).\end{aligned}$$

Set $\dot{\mathbf{P}}=(\dot{P}_1,\dots,\dot{P}_N)\in\mathcal{L}(\widehat{\mathcal{X}}\oplus\mathcal{X})^N$, with $\dot{P}_k:=\widehat{P}_k\oplus P_k,\ k=1,\dots,N$. Let us show that for $j=0,1,\dots$ the following non-commutative polynomial relations hold:

$$\begin{aligned}(z\mathbf{A})^j&=P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^j|\mathcal{X},&(5.14)\\(z\mathbf{A})^jz\mathbf{B}&=P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^j\dot{A}z\dot{\mathbf{P}}\dot{B},&(5.15)\\C(z\mathbf{A})^j&=\dot{C}(\dot{A}z\dot{\mathbf{P}}\dot{A})^j|\mathcal{X},&(5.16)\\C(z\mathbf{A})^jz\mathbf{B}&=\dot{C}(\dot{A}z\dot{\mathbf{P}}\dot{A})^j\dot{A}z\dot{\mathbf{P}}\dot{B}.&(5.17)\end{aligned}$$

For $j=0$ the equality (5.14) is trivial. For $j=1$ we have due to (5.12)

$$P_{\mathcal{X}}\dot{A}z\dot{\mathbf{P}}\dot{A}|\mathcal{X}=P_{\mathcal{X}}\hat{C}z\widehat{\mathbf{P}}\hat{B}|\mathcal{X}=z\mathbf{A},$$

i.e., (5.14) is fulfilled. Let us apply induction on j. Suppose that (5.14) holds for $j = k \in \mathbb{Z}_+ \setminus \{0\}$. Then, due to (5.12),

$$\begin{aligned}
(z\mathbf{A})^{k+1} &= z\mathbf{A}(z\mathbf{A})^k = P_{\mathcal{X}}\dot{A}z\dot{\mathbf{P}}\dot{A}P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^k|\mathcal{X} \\
&= P_{\mathcal{X}}\dot{A}z\dot{\mathbf{P}}\dot{A}(I_{\widehat{\mathcal{X}}\oplus\mathcal{X}} - P_{\widehat{\mathcal{X}}})(\dot{A}z\dot{\mathbf{P}}\dot{A})^k|\mathcal{X} \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X} - P_{\mathcal{X}}\dot{A}z\dot{\mathbf{P}}\dot{A}P_{\widehat{\mathcal{X}}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^k|\mathcal{X} \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X} - P_{\mathcal{X}}\hat{C}z\hat{\mathbf{P}}\hat{A}(\hat{A}z\hat{\mathbf{P}}\hat{A} + \hat{B}z\mathbf{P}P_{\mathcal{X}}\hat{C})P_{\widehat{\mathcal{X}}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k-1}|\mathcal{X} \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X} - P_{\mathcal{X}}(\hat{C}z\hat{\mathbf{P}}\hat{A})(\hat{A}z\hat{\mathbf{P}}\hat{A})P_{\widehat{\mathcal{X}}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k-1}|\mathcal{X} = \dots \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X} - P_{\mathcal{X}}(\hat{C}z\hat{\mathbf{P}}\hat{A})(\hat{A}z\hat{\mathbf{P}}\hat{A})^{k-1}P_{\widehat{\mathcal{X}}}(\dot{A}z\dot{\mathbf{P}}\dot{A})|\mathcal{X} \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X} - P_{\mathcal{X}}(\hat{C}z\hat{\mathbf{P}}\hat{A})(\hat{A}z\hat{\mathbf{P}}\hat{A})^{k-1}(\hat{A}z\hat{\mathbf{P}}\hat{B})|\mathcal{X} \\
&= P_{\mathcal{X}}(\dot{A}z\dot{\mathbf{P}}\dot{A})^{k+1}|\mathcal{X}.
\end{aligned}$$

Thus, (5.14) is fulfilled for all $j \in \mathbb{Z}_+$. Relations (5.15)–(5.17) are proved analogously. Set $\dot{\Sigma}^{\mathrm{FM,nc}} := (N; \dot{U}_z^{\mathrm{FM,nc}}; \dot{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, where $\dot{\mathcal{X}} := \widehat{\mathcal{X}} \oplus \mathcal{X}$

$$\dot{U}_z^{\mathrm{FM,nc}} := \begin{bmatrix} z\dot{\mathbf{A}} & z\dot{\mathbf{B}} \\ \dot{C} & \dot{D} \end{bmatrix},$$

with $\dot{\mathbf{A}} := (\dot{A}_1, \dots, \dot{A}_N) \in \mathcal{L}(\dot{\mathcal{X}})^N$, $\dot{\mathbf{B}} := (\dot{B}_1, \dots, \dot{B}_N) \in \mathcal{L}(\mathcal{U}, \dot{\mathcal{X}})^N$, $\dot{A}_k := \dot{A}\dot{P}_k\dot{A}$, $\dot{B}_k := \dot{A}\dot{P}_k\dot{B}$, $k = 1, \dots, N$. Then by Proposition 5.1, the system $\dot{\Sigma}^{\mathrm{FM,nc}}$ is a dilation of the system $\Sigma^{\mathrm{FM,nc}}$. Unlike at the parallel point in the proof of Theorem 4.5, it may not be the case that $\dot{\Sigma}^{\mathrm{FM,\,nc}}$ is $\mathcal{D}^N$-conservative (see Remark 5.6 below); we overcome this difficulty by introducing a second dilation as follows.

Let W be a unitary dilation of the contractive operator $\dot{A} = P_{\widehat{\mathcal{X}}\oplus\mathcal{X}}\dot{U}|(\widehat{\mathcal{X}} \oplus \mathcal{X})$ acting on a Hilbert space $\widetilde{\mathcal{X}} \supset \dot{\mathcal{X}} = \widehat{\mathcal{X}} \oplus \mathcal{X}$, which exists according to the Sz.-Nagy dilation theorem [28]. By the Sarason lemma [26, Lemma 0], there exist subspaces $\mathcal{D}$ and $\mathcal{D}_*$ in $\widetilde{\mathcal{X}}$ such that $\widetilde{\mathcal{X}} = \mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*$ and $W\mathcal{D} \subset \mathcal{D}$, $W^*\mathcal{D}_* \subset \mathcal{D}_*$. In other words, the operator W has the following operator-block matrix form:

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ 0 & \dot{A} & W_{23} \\ 0 & 0 & W_{33} \end{bmatrix} \in \mathcal{L}(\mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*).$$

Define also

$$z\ddot{\mathbf{P}} = \begin{bmatrix} z_1 I_{\mathcal{D}} & 0 & 0 \\ 0 & z\dot{\mathbf{P}} & 0 \\ 0 & 0 & z_1 I_{\mathcal{D}_*} \end{bmatrix} \in \mathcal{L}(\mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*)\left\langle z_1, \dots, z_N \right\rangle,$$

i.e.,

$$\ddot{P}_1 = \begin{bmatrix} I_{\mathcal{D}} & 0 & 0 \\ 0 & \dot{P}_1 & 0 \\ 0 & 0 & I_{\mathcal{D}_*} \end{bmatrix}, \quad \ddot{P}_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dot{P}_k & 0 \\ 0 & 0 & 0 \end{bmatrix} \ (k = 2, \dots, N),$$

$$\ddot{A} := \begin{bmatrix} I_{\mathcal{D}} & 0 & 0 \\ 0 & \dot{A} & 0 \\ 0 & 0 & I_{\mathcal{D}_*} \end{bmatrix} \in \mathcal{L}(\mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*), \quad \ddot{B} := \begin{bmatrix} 0 \\ \dot{B} \\ 0 \end{bmatrix} \in \mathcal{L}(\mathcal{U}, \mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*),$$

$$\ddot{C} := \begin{bmatrix} 0 & \dot{C} & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{D} \oplus \dot{\mathcal{X}} \oplus \mathcal{D}_*, \mathcal{Y}), \qquad \ddot{D} := \dot{D} = D.$$

Then

$$Wz\ddot{\mathbf{P}}\ddot{A} = \begin{bmatrix} z_1W_{11} & W_{12}z\dot{\mathbf{P}}\dot{A} & z_1W_{13} \\ 0 & z\dot{\mathbf{A}} & z_1W_{23} \\ 0 & 0 & z_1W_{33} \end{bmatrix}, \quad Wz\ddot{\mathbf{P}}\ddot{B} = \begin{bmatrix} W_{12}z\dot{\mathbf{P}}\dot{B} \\ z\dot{\mathbf{B}} \\ 0 \end{bmatrix}.$$

Set $\widetilde{\Sigma}^{\mathrm{FM,nc}} := (N; \widetilde{U}_z^{\mathrm{FM,nc}}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, where

$$\widetilde{U}_z^{\mathrm{FM,nc}} = \begin{bmatrix} z\widetilde{\mathbf{A}} & z\widetilde{\mathbf{B}} \\ \widetilde{C} & D \end{bmatrix},$$

$\widetilde{\mathbf{A}} := (\widetilde{A_1}, \ldots, \widetilde{A_N}) \in \mathcal{L}(\widetilde{\mathcal{X}})^N$, $\widetilde{A}_k := W\ddot{P}_k\ddot{A}$ $(k = 1, \ldots, N)$, $\widetilde{\mathbf{B}} := (\widetilde{B_1}, \ldots, \widetilde{B_N}) \in \mathcal{L}(\mathcal{U}, \widetilde{\mathcal{X}})^N$, $\widetilde{B_k} := W\ddot{P}_k\ddot{B}$ $(k = 1, \ldots, N)$, $\widetilde{C} := \ddot{C}$. Then it is easy to check that, with the choice of subspaces $\mathcal{D}$ and $\mathcal{D}_*$ in $\widetilde{\mathcal{X}}$, the system $\widetilde{\Sigma}^{\mathrm{FM,nc}}$ is a uniform dilation of the system $\dot{\Sigma}^{\mathrm{FM,nc}}$. Since a dilation of a dilation is again a dilation, the system $\widetilde{\Sigma}^{\mathrm{FM,nc}}$ is a dilation of the system $\Sigma^{\mathrm{FM,nc}}$. Moreover, since the operator $\dot{U}$ is unitary, so is

$$\ddot{U} = \begin{bmatrix} \ddot{A} & \ddot{B} \\ \ddot{C} & \ddot{D} \end{bmatrix}.$$

If $\mathbf{V} \in \mathcal{U}^N \cap \mathcal{L}(\mathcal{H})^N$, with some Hilbert space $\mathcal{H}$, then we have the factorization

$$\widetilde{U}_{\mathbf{V}}^{\mathrm{FM}} = \begin{bmatrix} (W \otimes I_{\mathcal{H}})(\sum_{k=1}^N \widetilde{P}_k \otimes V_k) & 0 \\ 0 & I_{\mathcal{Y}} \otimes I_{\mathcal{H}} \end{bmatrix} \ddot{U}$$

where the left factor is also unitary. Thus $\widetilde{\Sigma}^{\mathrm{FM,nc}}$ is a $\mathcal{D}^N$-conservative FM system and $\widetilde{\Sigma}^{\mathrm{FM,nc}}$ is a $\mathcal{D}^N$-conservative dilation of the system $\Sigma^{\mathrm{FM,nc}}$. By Proposition 5.4, this dilation is uniform. □

Remark 5.6. Our proof of Theorem 5.5 is based on the close relationship between a $\mathcal{D}^N$-conservative realization of the linear function $U_z^{\mathrm{FM,nc}}$ and a certain dilation of the corresponding non-commutative FM system $\Sigma^{\mathrm{FM,nc}}$, i.e., we use here the same idea as in our proof of Theorem 4.5 on conservative dilation of a noncommutative KV system (let us remark that this idea was used for the first time in [20] for the proof of the conservative dilation theorem for commutative KV systems). However, the proof of Theorem 5.5 appears to be more complicated in that it requires two (rather than a single) dilation. The necessity for this extra dilation can be explained as follows.

In the proof of Theorem 4.5 we consider a conservative realization $\dot{\Sigma}^{\mathrm{KV,nc}}$ of the linear function $U_z^{\mathrm{KV,nc}}$ and then rearrange the underlying spaces, so that the state space $\mathcal{X}$ of the system $\Sigma^{\mathrm{KV,nc}}$ which was a part of both input and output spaces of the system $\dot{\Sigma}^{\mathrm{KV,nc}}$ becomes a part of the state space of the new system $\tilde{\Sigma}^{\mathrm{KV,nc}}$ and the linear function $\tilde{U}_z^{\mathrm{KV,nc}}$ coincides with $\dot{U}_z^{\mathrm{KV,nc}}$. It turns out that $\tilde{\Sigma}^{\mathrm{KV,nc}}$ is a conservative dilation of $\Sigma^{\mathrm{KV,nc}}$. In the proof of Theorem 5.5 we first consider a conservative GR realization $\hat{\Sigma}^{\mathrm{GR,nc}}$ of the linear function $U_z^{\mathrm{FM,nc}}$, which can then be viewed also as a $\mathcal{D}^N$-conservative FM realization of the linear function $U_z^{\mathrm{FM,nc}}$. In the case of FM systems, the linear function $U_z^{\mathrm{FM,nc}}$ is not necessarily homogeneous; moreover, its "A and B blocks" are homogeneous, while its "C and D blocks" are constants. The latter results in more complicated relations (involving products of the contractions $\hat{C}$ and $\hat{B}$ and some projection operators) between the blocks of $U_z^{\mathrm{FM,nc}}$ and the blocks of $\hat{U}_z^{\mathrm{GR,nc}}$ (see (5.12)) than the ones between the blocks of $U_z^{\mathrm{KV,nc}}$ and $\dot{U}_z^{\mathrm{KV,nc}}$ (see (4.17) and the line above it) in the proof of Theorem

4.5 for the case of KV systems. Nevertheless, the above mentioned relations allow us to construct a $\mathcal{D}^N$-dissipative FM system $\dot{\Sigma}^{\rm FM,nc}$ which is a dilation of $\Sigma^{\rm FM,nc}$ but which is not necessarily $\mathcal{D}^N$-conservative. This limitation then necessitates the construction of a second, $\mathcal{D}^N$-conservative, dilation to complete the proof.

Remark 5.7. In the case where the feedthrough operator D of the (commutative or non-commutative) FM system $\Sigma^{\rm FM}$ is zero, the connection between FM and KV systems enables one to deduce Theorem 5.5 as a consequence of Theorem 4.5 (the authors are thankful to an anonymous referee to pointing this out). This connection was established in [11] for commutative $\mathbb{D}^N$-conservative FM and KV systems (the condition $D = 0$ on a FM system is essential!); however, the same connection exists for non-commutative $\mathcal{D}^N$-conservative FM and KV systems, and for commutative or non-commutative polydisk-dissipative FM and KV systems. Let us also remark that the above connection can also be used to deduce Theorem 4.5 from Theorem 5.5; however the latter procedure leads to no logical shortcuts since our direct proof of Theorem 4.5 is simpler than the one of Theorem 5.5.

5.2. **The commutative setting.** We will say that the commutative FM system $\widetilde{\Sigma}^{\rm FM,c} = (N; \widetilde{U}_z^{\rm FM,c}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a *dilation of the commutative FM system* $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ if for every fixed $z \in \mathbb{C}^N$ the 1D system $\widetilde{\Sigma}_z^{\rm FM,c} := (1; z\widetilde{\mathbf{A}}, z\widetilde{\mathbf{B}}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a dilation of the corresponding 1D system $\Sigma_z^{\rm FM,c} := (1; z\mathbf{A}, z\mathbf{B}, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, i.e., for each $z \in \mathbb{C}^N$ there exist subspaces $\mathcal{D}_z$ and $\mathcal{D}_{*,z}$ in $\widetilde{\mathcal{X}}$ such that

$$\widetilde{\mathcal{X}} = \mathcal{D}_z \oplus \mathcal{X} \oplus \mathcal{D}_{*,z}, \tag{5.18}$$

$$z\widetilde{\mathbf{A}}\mathcal{D}_z \subset \mathcal{D}_z,\ \widetilde{C}\mathcal{D}_z = \{0\},\ (z\widetilde{\mathbf{A}})^*\mathcal{D}_{*,z} \subset \mathcal{D}_{*,z},\ (z\widetilde{\mathbf{B}})^*\mathcal{D}_{*,z} = \{0\}, \tag{5.19}$$

$$z\mathbf{A} = P_{\mathcal{X}}(z\widetilde{\mathbf{A}})|\mathcal{X},\ z\mathbf{B} = P_{\mathcal{X}}(z\widetilde{\mathbf{B}}),\ C = \widetilde{C}|\mathcal{X},\ D = \widetilde{D}. \tag{5.20}$$

From Lemma 1.1 we obtain the following equivalent reformulation of this definition.

Proposition 5.8. *The system* $\widetilde{\Sigma}^{\rm FM,c} = (N; \widetilde{U}_z^{\rm FM,c}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}} \supset \mathcal{X}$, $\widetilde{D} = D$, *and for all* $z \in \mathbb{C}^N$ *and* $j \in \mathbb{Z}_+$ *the following equalities hold:*

$$\begin{array}{ll} (z\mathbf{A})^j = P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & (z\mathbf{A})^j z\mathbf{B} = P_{\mathcal{X}}(z\widetilde{\mathbf{A}})^j z\widetilde{\mathbf{B}}, \\ C(z\mathbf{A})^j = \widetilde{C}(z\widetilde{\mathbf{A}})^j|\mathcal{X}, & C(z\mathbf{A})^j z\mathbf{B} = \widetilde{C}(z\widetilde{\mathbf{A}})^j z\widetilde{\mathbf{B}}. \end{array} \tag{5.21}$$

Rewriting the equalities for commutative polynomials in (5.21) as the equalities for their coefficients, we obtain the following commutative FM-system version of the Sarason lemma.

Proposition 5.9. *The system* $\widetilde{\Sigma}^{\rm FM,c} = (N; \widetilde{U}_z^{\rm FM,c}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *is a dilation of the system* $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *if and only if* $\widetilde{\mathcal{X}} \supset \mathcal{X}$, $\widetilde{D} = D$, *and the following equalities hold:*

$$\begin{array}{cc} \forall s \in \mathbb{Z}_+^N & \mathbf{A}^s = P_{\mathcal{X}}\widetilde{\mathbf{A}}^s|\mathcal{X}, \\ \forall s \in \mathbb{Z}_+^N \setminus \{0\} & (\mathbf{A}\sharp\mathbf{B})^s = P_{\mathcal{X}}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^s, \\ \forall s \in \mathbb{Z}_+^N & C\mathbf{A}^s = \widetilde{C}\widetilde{\mathbf{A}}^s|\mathcal{X}, \\ \forall s \in \mathbb{Z}_+^N \setminus \{0\} & C(\mathbf{A}\sharp\mathbf{B})^s = \widetilde{C}(\widetilde{\mathbf{A}}\sharp\widetilde{\mathbf{B}})^s. \end{array} \tag{5.22}$$

As a corollary of the last of equalities (5.21) in Proposition 5.8 we obtain the following statement.

Proposition 5.10. *The transfer functions $T_{\Sigma^{\rm FM,c}}$ and $T_{\widetilde{\Sigma}^{\rm FM,c}}$ of a system $\Sigma^{\rm FM,c}$ and of its dilation $\widetilde{\Sigma}^{\rm FM,c}$ coincide.*

Theorem 5.11. *The $\mathbb{D}^N$-dissipative FM system $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ has a $\mathbb{D}^N$-conservative dilation if and only if the linear polynomial*

$$U_z^{\rm FM,c} = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})[z_1,\dots,z_N] \tag{5.23}$$

belongs to the class $\mathcal{SA}_N(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$.

Proof. Let the $\mathbb{D}^N$-dissipative system $\Sigma^{\rm FM,c} = (N; U_z^{\rm FM,c}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ have a $\mathbb{D}^N$-conservative dilation $\widetilde{\Sigma}^{\rm FM,c} = (N; \widetilde{U}_z^{\rm FM}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$. Then for each $\zeta \in \mathbb{T}^N$ the operator

$$\widetilde{U}_\zeta^{\rm FM,c} = \begin{bmatrix} \zeta\widetilde{\mathbf{A}} & \zeta\widetilde{\mathbf{B}} \\ \widetilde{C} & D \end{bmatrix} \in \mathcal{L}(\widetilde{\mathcal{X}}\oplus\mathcal{U}, \widetilde{\mathcal{X}}\oplus\mathcal{Y})$$

is unitary. As was observed in Section 3, the non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc} = (N; \widetilde{U}_z^{\rm FM,nc}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is then $\mathcal{D}^N$-conservative, i.e., for every $\mathbf{V} \in \mathcal{U}^N$ the operator

$$\widetilde{U}_{\mathbf{V}}^{\rm FM,nc} = \begin{bmatrix} \widetilde{\mathbf{A}}\otimes\mathbf{V} & \widetilde{\mathbf{B}}\otimes\mathbf{V} \\ \widetilde{C}\otimes I_{\mathcal{H}} & D\otimes I_{\mathcal{H}} \end{bmatrix}$$

is unitary. Since an arbitrary N-tuple $\mathbf{T} = (T_1,\dots,T_N)$ of (not necessarily strict and not necessarily commuting) contractions on a Hilbert space, say $\mathcal{H}$, has a unitary dilation $\mathbf{V} = (V_1,\dots,V_N)$ (see [29]), we have $\|\widetilde{U}_{\mathbf{T}}^{\rm FM,nc}\| \le \|\widetilde{U}_{\mathbf{V}}^{\rm FM,nc}\| = 1$, and thus

$$\begin{aligned} \left(U_{\mathbf{T}}^{\rm FM,nc}\right)^* U_{\mathbf{T}}^{\rm FM,nc} &= P_{(\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}}\left(\widetilde{U}_{\mathbf{T}}^{\rm FM,nc}\right)^* P_{(\mathcal{X}\oplus\mathcal{Y})\otimes\mathcal{H}}\widetilde{U}_{\mathbf{T}}^{\rm FM,nc}\big|((\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}) \\ &\le P_{(\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}}\left(\widetilde{U}_{\mathbf{T}}^{\rm FM,nc}\right)^* \widetilde{U}_{\mathbf{T}}^{\rm FM,nc}\big|((\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}) \\ &\le I_{(\mathcal{X}\oplus\mathcal{U})\otimes\mathcal{H}}, \end{aligned}$$

we conclude that $U_z^{\rm FM,nc} \in \mathcal{SA}_N^{nc}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$, and then $U_z^{\rm FM,c} \in \mathcal{SA}_N(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$.

Conversely, suppose that $U_z^{\rm FM,c} \in \mathcal{SA}_N(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$. Let us show that the associated linear non-commutative polynomial $U_z^{\rm FM,nc}$ belongs to the class $\mathcal{SA}_N^{nc}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$. Indeed, if $\mathbf{T} = (T_1,\dots,T_N) \in \mathcal{D}_{\rm matr}^N \cap (\mathbb{C}^{n\times n})^N$ for some $n \in \mathbb{Z}_+ \setminus \{0\}$ then the N-tuple $\widetilde{\mathbf{T}} = (\widetilde{T_1},\dots,\widetilde{T_N}) \in (\mathbb{C}^{2n\times 2n})^N$, where

$$\widetilde{T_k} := \begin{bmatrix} 0 & T_k \\ 0 & 0 \end{bmatrix}, \qquad k = 1,\dots,N,$$

consists of commuting strictly contractive matrices (note that $\widetilde{T_k}\widetilde{T_j} = 0,\ k,j = 1,\dots,N$). Since

$$\left\|U_{\mathbf{T}}^{\rm FM,nc}\right\| = \left\|P_{(\mathcal{X}\oplus\mathcal{Y})\otimes(\mathbb{C}^n\oplus\{0\})}U_{\widetilde{\mathbf{T}}}^{\rm FM,c}\big|((\mathcal{X}\oplus\mathcal{U})\otimes(\{0\}\oplus\mathbb{C}^n))\right\| \le \left\|U_{\widetilde{\mathbf{T}}}^{\rm FM,c}\right\| \le 1,$$

we obtain $U_z^{\rm FM,nc} \in \mathcal{SA}_N^{nc}(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$. It is clear that for every N-tuple of (not necessarily strict) contractions $\mathbf{T}$ one has $\|U_{\mathbf{T}}^{\rm FM,nc}\| \le 1$, and so this holds for every N-tuple of unitary operators. The latter means that the associated non-commutative FM system $\Sigma^{\rm FM,nc} = (N; U_z^{\rm FM,nc}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is $\mathcal{D}^N$-dissipative. Then according to Theorem 5.5 there exists a non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc} = (N; \widetilde{U}_z^{\rm FM,nc}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ which is a (uniform) $\mathcal{D}^N$-conservative dilation of $\Sigma^{\rm FM,nc}$. One can easily observe now that the corresponding commutative FM system $\widetilde{\Sigma}^{\rm FM,c} = (N;\widetilde{U}_z^{\rm FM,c};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a $\mathbb{D}^N$-conservative dilation of $\Sigma^{\rm FM,c}$. □

Let us remark that for $N\geq 3$ not all $\mathbb{D}^N$-dissipative FM systems have $\mathbb{D}^N$-conservative dilations. Indeed, according to [23], for every $N\geq 3$ one can find an N-tuple of operators $\mathbf{A}=(A_1,\ldots,A_N)$ on a Hilbert space, say $\mathcal{X}$, such that $A_z := z\mathbf{A}\in\mathcal{B}_N(\mathcal{X})\setminus\mathcal{SA}_N(\mathcal{X})$. In other words, the commutative FM system $\Sigma^{\rm FM,c} = (N;U_z^{\rm FM,c}=z\mathbf{A};\mathcal{X},\{0\},\{0\})$ is $\mathbb{D}^N$-dissipative, however the corresponding linear function $\mathbf{A}_z$ doesn't belong to the class $\mathcal{SA}_N(\mathcal{X})$. By Theorem 5.11, the system $\Sigma^{\rm FM,c}$ has no $\mathbb{D}^N$-conservative dilation.

Let the system $\widetilde{\Sigma}^{\rm FM,c}=(N;\widetilde{U}_z^{\rm FM,c};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ be a dilation of the system $\Sigma^{\rm FM,c} = (N;U_z^{\rm FM,c};\mathcal{X},\mathcal{U},\mathcal{Y})$. We will say that such a dilation is *uniform* if the subspaces $\mathcal{D}_z$ are independent of $z\in\mathbb{C}^N$, i.e., $\mathcal{D}_z=\mathcal{D},\ z\in\mathbb{C}^N$, or equivalently, subspaces $\mathcal{D}_{*,z}$ are independent of $z\in\mathbb{C}^N$, i.e., $\mathcal{D}_{*,z}=\mathcal{D},\ z\in\mathbb{C}^N$. It is easy to see that the commutative FM system $\widetilde{\Sigma}^{\rm FM,c}=(N;\widetilde{U}_z^{\rm FM,c};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a uniform dilation of the commutative FM system $\Sigma^{\rm FM,c}=(N;U_z^{\rm FM,c};\mathcal{X},\mathcal{U},\mathcal{Y})$ if and only if the corresponding non-commutative FM system $\widetilde{\Sigma}^{\rm FM,nc}=(N;\widetilde{U}_z^{\rm FM,nc};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a uniform dilation of the corresponding non-commutative FM system $\Sigma^{\rm FM,nc}=(N;U_z^{\rm FM,nc};\mathcal{X},\mathcal{U},\mathcal{Y})$.

The following theorem is an improvement of Theorem 5.11.

Theorem 5.12. *The $\mathbb{D}^N$-dissipative system $\Sigma^{\rm FM,c}=(N;U_z^{\rm FM,c};\mathcal{X},\mathcal{U},\mathcal{Y})$ has a uniform $\mathbb{D}^N$-conservative dilation if and only if the linear polynomial* (5.23) *belongs to the class $\mathcal{SA}_N(\mathcal{X}\oplus\mathcal{U},\mathcal{X}\oplus\mathcal{Y})$.*

Proof. The necessity part follows from Theorem 5.11. For the proof of the sufficiency part, observe that the $\mathcal{D}^N$-conservative dilation $\widetilde{\Sigma}^{\rm FM,nc}$ of the system $\Sigma^{\rm FM,nc}$ in Theorem 5.11 is uniform. Thus, in view of the remark preceding this Theorem, the corresponding $\mathbb{D}^N$-conservative dilation $\widetilde{\Sigma}^{\rm FM,c}$ of the system $\Sigma^{\rm FM,c}$ is also uniform. □

Let us remark that in the commutative case not every dilation of a FM system is uniform (in contrast with the non-commutative case, see Proposition 5.4). Moreover, not every $\mathbb{D}^N$-conservative dilation is uniform (which implies that Theorem 5.12 is an improvement of Theorem 5.11). To show this, we may use Example 4.12 and the observation that for the case where the input and output spaces are zero (i.e., $\mathcal{U}=\mathcal{Y}=\{0\}$) the notions of KV system and of FM system coincide, both in the commutative and in the non-commutative settings.

6. Dilations of BGM systems

6.1. **The non-commutative setting.** We will say that the non-commutative BGM system $\widetilde{\Sigma}^{\rm BGM,nc}=(G;\widetilde{U};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ is a *dilation* of the non-commutative BGM system $\Sigma^{\rm BGM,nc}=(G;U;\mathcal{X},\mathcal{U},\mathcal{Y})$ if, for each $n\in\mathbb{Z}_+\setminus\{0\}$, $\mathbf{Z}=(Z_{e_1},\ldots,Z_{e_N})\in(\mathbb{C}^{n\times n})^N$ and $s\in\{s_1,\ldots,s_{n_S}\}$ there exist subspaces $\mathcal{D}_{\mathbf{Z},s}$ and $\mathcal{D}_{*,\mathbf{Z},s}$ in $\widetilde{\mathcal{X}}_s\otimes\mathbb{C}^n$

such that

$$\widetilde{\mathcal{X}}_s \otimes \mathbb{C}^n = \mathcal{D}_{\mathbf{Z},s} \oplus (\mathcal{X}_s \otimes \mathbb{C}^n) \oplus \mathcal{D}_{*,\mathbf{Z},s} \text{ for } s = s_1, \dots, s_{n_S}, \tag{6.1}$$

$$\begin{gathered}(\widetilde{A}_{r(e),s} \otimes Z_e)\mathcal{D}_{\mathbf{Z},s} \subset \mathcal{D}_{\mathbf{Z},s(e)}, \qquad (\widetilde{C}_s \otimes I_n)\mathcal{D}_{\mathbf{Z},s} = \{0\},\\ (\widetilde{A}_{r(e),s} \otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z},s(e)} \subset \mathcal{D}_{*,\mathbf{Z},s}, \quad (\widetilde{B}_{r(e)} \otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z},s(e)} = \{0\},\\ \text{for } e = e_1, \dots, e_N \text{ and } s = s_1, \dots, s_{n_S},\end{gathered} \tag{6.2}$$

$$\begin{gathered}A_{r,s} = P_{\mathcal{X}_r}\widetilde{A}_{r,s}|_{\mathcal{X}_s}, \qquad B_r = P_{\mathcal{X}_r}\widetilde{B}_r, \qquad C_s = \widetilde{C}_s|_{\mathcal{X}_s}, \qquad D = \tilde{D},\\ \text{for } r = r_1, \dots, r_{n_R} \text{ and } s = s_1, \dots, s_{n_S}.\end{gathered} \tag{6.3}$$

In particular, we have the following special cases of this definition.

Non-commutative FM systems. In this case $E = \{e_1, \dots, e_N\}$, $S = \{s_1\}$, $R = \{r_1, \dots, r_N\}$, $A = \mathrm{col}_{\mathrm{k=1,\dots,n}}[\mathrm{A_k}]$, $\mathrm{B} = \mathrm{col}_{\mathrm{k=1,\dots,n}}[\mathrm{B_k}]$. Thus (6.1) turns into

$$\widetilde{\mathcal{X}} \otimes \mathbb{C}^n = D_{\mathbf{Z}} \oplus (\mathcal{X} \otimes \mathbb{C}^n) \oplus D_{*,\mathbf{Z}}, \tag{6.4}$$

(6.2) turns into

$$\begin{matrix}(\widetilde{A}_k \otimes Z_k)\mathcal{D}_{\mathbf{Z}} \subset \mathcal{D}_{\mathbf{Z}}, & (\widetilde{C} \otimes I_n)\mathcal{D}_{\mathbf{Z}} = \{0\}\\ (\widetilde{A}_k \otimes Z_k)^*\mathcal{D}_{*,\mathbf{Z}} \subset \mathcal{D}_{*,\mathbf{Z}}, & (\widetilde{B}_k \otimes Z_k)^*\mathcal{D}_{*,\mathbf{Z}} = \{0\},\end{matrix} \quad k = 1, \dots, N, \tag{6.5}$$

and (6.3) turns into

$$A_k = P_{\mathcal{X}}\widetilde{A}_k|_{\mathcal{X}}, \quad B_k = P_{\mathcal{X}}\widetilde{B}_k, \quad C = \widetilde{C}|_{\mathcal{X}}, \quad D = \tilde{D}, \quad k = 1, \dots, N, \tag{6.6}$$

and altogether they are equivalent to the definition of dilation for non-commutative FM systems given in Section 5.1. Indeed, (6.4) coincides with (5.1), (6.5) implies (5.2), and (6.6) implies (5.3). On the other hand, by Proposition 5.4, any dilation of a non-commutative FM system is uniform, and choosing $\mathcal{D}_{\mathbf{Z}} = \mathcal{D} \otimes \mathbb{C}^n$ and $\mathcal{D}_{*,\mathbf{Z}} = \mathcal{D}_* \otimes \mathbb{C}^n$ as in the definition of uniform dilation, we obtain that (5.9)–(5.11) imply (6.4)–(6.6).

Non-commutative GR systems. Here $E = \{e_1, \dots, e_N\}$, $S = \{s_1, \dots, s_N\}$, $R = \{r_1, \dots, r_N\}$. Then (6.1)–(6.3) turn into

$$\widetilde{\mathcal{X}}_k \otimes \mathbb{C}^n = \mathcal{D}_{\mathbf{Z},k} \oplus (\mathcal{X}_k \otimes \mathbb{C}^n) \oplus \mathcal{D}_{*,\mathbf{Z},k}, \quad k = 1, \dots, N, \tag{6.7}$$

$$\begin{matrix}(\widetilde{A}_{kj} \otimes Z_k)\mathcal{D}_{\mathbf{Z},j} \subset \mathcal{D}_{\mathbf{Z},k}, & (\widetilde{C}_j \otimes I_n)\mathcal{D}_{\mathbf{Z},j} = \{0\},\\ (\widetilde{A}_{kj} \otimes Z_j)^*\mathcal{D}_{*,\mathbf{Z},k} \subset \mathcal{D}_{*,\mathbf{Z},j}, & (\widetilde{B}_k \otimes Z_k)^*\mathcal{D}_{*,\mathbf{Z},k} = \{0\},\end{matrix} \quad k, j = 1, \dots, N, \tag{6.8}$$

$$A_{kj} = P_{\mathcal{X}_k}\widetilde{A}_{kj}|_{\mathcal{X}_j}, \quad B_k = P_{\mathcal{X}_k}\widetilde{B}_k, \quad C_j = \widetilde{C}_j|_{\mathcal{X}_j}, \quad k, j = 1, \dots, N. \tag{6.9}$$

Remark 6.1. Let us show that the dilation $\tilde{\Sigma}^{\mathrm{BGM,nc}} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ of a non-commutative BGM system $\Sigma^{\mathrm{BGM,nc}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, when these two systems are considered as non-commutative FM systems (see Section 2.4), is in particular a dilation of a non-commutative FM system in the sense of Section 5.1. Set $\mathcal{D}_{\mathbf{Z}} := \bigoplus_{s\in S} \mathcal{D}_{\mathbf{Z},s}$, $\mathcal{D}_{*,\mathbf{Z}} := \bigoplus_{s\in S} \mathcal{D}_{*,\mathbf{Z},s}$, $A_k := I_{\Sigma,e_k}A \in \mathcal{L}(\bigoplus_{s\in S} \mathcal{X}_s)$, $B_k = I_{\Sigma,e_k}B \in$

$\mathcal{L}(\mathcal{U},\bigoplus_{s\in S}\mathcal{X}_s)$, $k=1,\dots,N$. Then (5.1) and (5.3) are obvious. We see also that

$$\begin{aligned}(I_{\widetilde{\Sigma},e}\widetilde{A}\otimes Z_e)\mathcal{D}_{\mathbf{Z},s} &= \left(\begin{bmatrix} 0 & \dots & 0\\ \vdots & & \vdots\\ \widetilde{A}_{r(e),s_1} & \dots & \widetilde{A}_{r(e),s_{n_S}}\\ \vdots & & \vdots\\ 0 & \dots & 0\end{bmatrix}\otimes Z_e\right)\mathcal{D}_{\mathbf{Z},s}\\ &= (\widetilde{A}_{r(e),s}\otimes Z_e)\mathcal{D}_{\mathbf{Z},s}\subset\mathcal{D}_{\mathbf{Z},s(e)}\subset\mathcal{D}_{\mathbf{Z}},\end{aligned}$$

where in the matrix above only the $s(e)$-th block row is non-zero, and we consider the spaces $\mathcal{D}_{\mathbf{Z},s}$ and $\mathcal{D}_{\mathbf{Z},s(e)}$ as the subspaces of the s-th and $s(e)$-th components $\mathcal{X}_s$ and $\mathcal{X}_{s(e)}$ in the orthogonal sum $\bigoplus_{s\in S}\mathcal{X}_s$, respectively. Therefore,

$$(\widetilde{\mathbf{A}}\otimes\mathbf{Z})\mathcal{D}_{\mathbf{Z}}=\sum_{e\in E}(I_{\widetilde{\Sigma},e}\widetilde{A}\otimes Z_e)\bigoplus_{s\in S}\mathcal{D}_{\mathbf{Z},s}\subset\mathcal{D}_{\mathbf{Z}}.$$

Since $(\widetilde{C}_s\otimes I_n)\mathcal{D}_{\mathbf{Z},s}=\{0\}$, we have

$$(\widetilde{C}\otimes I_n)\mathcal{D}_{\mathbf{Z}}=\sum_{s\in S}(\widetilde{C}_s\otimes I_n)\mathcal{D}_{\mathbf{Z},s}=\{0\}.$$

We see also that

$$\begin{aligned}(I_{\widetilde{\Sigma},e}\widetilde{A}\otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z},s(e)} &= \left(\begin{bmatrix} 0 & \dots & (\widetilde{A}_{r(e),s_1})^* & \dots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \dots & (\widetilde{A}_{r(e),s_{n_S}})^* & \dots & 0\end{bmatrix}\otimes Z_e^*\right)\mathcal{D}_{*,\mathbf{Z},s(e)}\\ &= \bigoplus_{j=1}^{n_S}(\widetilde{A}_{r(e),s_j}\otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z},s(e)}\subset\bigoplus_{j=1}^{n_S}\mathcal{D}_{*,\mathbf{Z},s_j}=\mathcal{D}_{*,\mathbf{Z}},\end{aligned}$$

where in the matrix above only the $s(e)$-th block column is non-zero. Therefore,

$$(\widetilde{\mathbf{A}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}=\sum_{e\in E}(I_{\widetilde{\Sigma},e}\widetilde{A}\otimes Z_e)^*\bigoplus_{s\in S}\mathcal{D}_{*,\mathbf{Z},s}=\sum_{e\in E}(I_{\widetilde{\Sigma},e}\widetilde{A}\otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z},s(e)}\subset\mathcal{D}_{*,\mathbf{Z}}.$$

Since

$$\begin{aligned}(I_{\widetilde{\Sigma},e}\widetilde{B}\otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z}} &= \left(\begin{bmatrix} 0 & \dots & (\widetilde{B}_{r(e)})^* & \dots & 0\end{bmatrix}\otimes Z_e^*\right)\bigoplus_{s\in S}\mathcal{D}_{*,\mathbf{Z},s}\\ &= ((\widetilde{B}_{r(e)})^*\otimes Z_e^*)\mathcal{D}_{*,\mathbf{Z},s(e)}=\{0\},\end{aligned}$$

where in the row matrix above only the $s(e)$-th block entry is non-zero, we get

$$(\widetilde{\mathbf{B}}\otimes\mathbf{Z})^*\mathcal{D}_{*,\mathbf{Z}}=\sum_{e\in E}(I_{\widetilde{\Sigma},e}\widetilde{B}\otimes Z_e)^*\mathcal{D}_{*,\mathbf{Z}}=\{0\}.$$

Finally, all the relations in (5.2) are fulfilled.

Remark 6.1 together with Proposition 5.2 imply the following.

Proposition 6.2. *The transfer functions $T_{\Sigma^{\rm BGM,nc}}$ and $T_{\widetilde{\Sigma}^{\rm BGM,nc}}$ of a system $\Sigma^{\rm BGM,nc}=(G;U;\mathcal{X},\mathcal{U},\mathcal{Y})$ and of its dilation $\widetilde{\Sigma}^{\rm BGM,nc}=(G;\widetilde{U};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y})$ coincide.*

Let the non-commutative BGM system $\widetilde{\Sigma}^{\rm BGM,nc} = (G; \widetilde{U}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ be a dilation of the non-commutative BGM system $\Sigma^{\rm BGM,nc} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. We will say that such a dilation is *uniform* if the subspaces $\mathcal{D}_{\mathbf{Z},s}$ and $\mathcal{D}_{*,\mathbf{Z},s}$ for $s = s_1, \ldots, s_{n_S}$ from the equalities (6.1) and (6.2) are independent of $\mathbf{Z}$ and have the form $\mathcal{D}_{\mathbf{Z},s} = \mathcal{D}_s \otimes \mathbb{C}^n$ and $\mathcal{D}_{*,\mathbf{Z},s} = \mathcal{D}_{*,s} \otimes \mathbb{C}^n$. Thus we say that $\widetilde{\Sigma}^{\rm BGM,nc}$ is a *uniform dilation* of $\Sigma^{\rm BGM,nc}$ if there exist subspaces $\mathcal{D}_s$ and $\mathcal{D}_{*,s}$ in $\widetilde{\mathcal{X}}_s$ such that

$$\widetilde{\mathcal{X}}_s = \mathcal{D}_s \oplus \mathcal{X}_s \oplus \mathcal{D}_{*,s} \text{ for } s = s_1, \ldots, s_{n_S}, \tag{6.10}$$

$$\begin{aligned} &\widetilde{A}_{r(e),s} \mathcal{D}_s \subset \mathcal{D}_{s(e)}, \qquad \widetilde{C}_s \mathcal{D}_s = \{0\}, \\ &\widetilde{A}^*_{r(e),s} \mathcal{D}_{*,s(e)} \subset \mathcal{D}_{*,s}, \quad \widetilde{B}^*_{r(e)} \mathcal{D}_{*,s(e)} = \{0\}, \\ &\quad \text{for } e = e_1, \ldots, e_N, \ s = s_1, \ldots, s_{n_S}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} &A_{r,s} = P_{\mathcal{X}_r} \widetilde{A}_{r,s}|_{\mathcal{X}_s}, \quad B_r = P_{\mathcal{X}_r} \widetilde{B}_r, \quad C_s = \widetilde{C}_s|_{\mathcal{X}_s}, \quad D = \tilde{D} \\ &\quad \text{for } r = r_1, \ldots, r_{n_R}, \ s = s_1, \ldots, s_{n_S}. \end{aligned} \tag{6.12}$$

It is easy to see that this definition agrees with the definition of uniform dilation given in Section 5.1 for the particular case of non-commutative FM systems. For the particular case of non-commutative GR systems, (6.10)–(6.12) turn into

$$\widetilde{\mathcal{X}}_k = \mathcal{D}_k \oplus \mathcal{X}_k \oplus \mathcal{D}_{*,k} \text{ for } k = 1, \ldots, N, \tag{6.13}$$

$$\begin{matrix} \widetilde{A}_{kj} \mathcal{D}_j \subset \mathcal{D}_k, & \widetilde{C}_j \mathcal{D}_j = \{0\}, \\ \widetilde{A}^*_{kj} \mathcal{D}_{*,k} \subset \mathcal{D}_{*,j}, & \widetilde{B}^*_k \mathcal{D}_{*,k} = \{0\}, \end{matrix} \ , \ k, j = 1, \ldots, N, \tag{6.14}$$

$$A_{kj} = P_{\mathcal{X}_k} \widetilde{A}_{kj}|_{\mathcal{X}_j}, \ B_k = P_{\mathcal{X}_k} \widetilde{B}_k, \ C_j = \widetilde{C}_j|_{\mathcal{X}_j}, \ D = \tilde{D}, \quad k, j = 1, \ldots, N. \tag{6.15}$$

Given a BGM system $\Sigma^{\rm BGM} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, let us introduce, adapting the notation from [2], the following notations: for $w = e_{i_1} \cdots e_{i_m}$ a word in the edges of the graph G, with $m = 2, 3, \ldots$, set

$$\begin{aligned} A^w &:= A_{r(e_{i_1}), s(e_{i_2})} A_{r(e_{i_2}), s(e_{i_3})} \cdots A_{r(e_{i_{m-1}}), s(e_{i_m})}, \quad (C\flat A)^w := C_{s(e_{i_1})} A^w, \\ &\qquad (A\sharp B)^w := A^w B_{r(e_{i_m})}, \quad (C\flat A\sharp B)^w = C_{s(e_{i_1})} A^w B_{r(e_{i_m})}, \end{aligned}$$

and also

$$A^{e_k} := I_{\mathcal{X}_{r(e_k)}} = I_{\mathcal{X}_{s(e_k)}}, \ (C\flat A)^{e_k} := C_{s(e_k)}, \ (A\sharp B)^{e_k} := B_{r(e_k)}, \ k = 1, \ldots, N.$$

Proposition 6.3. *For non-commutative BGM systems $\Sigma^{\rm BGM,nc} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\widetilde{\Sigma}^{\rm BGM,nc} = (G; \widetilde{U}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ the following statements are equivalent:*

- **(i):** *$\widetilde{\Sigma}^{\rm BGM,nc}$ is a dilation of $\Sigma^{\rm BGM,nc}$;*
- **(ii):** *$\widetilde{\Sigma}^{\rm BGM,nc}$ is a uniform dilation of $\Sigma^{\rm BGM,nc}$;*
- **(iii):** *$\widetilde{\mathcal{X}}_p \supset \mathcal{X}_p$ for $p = p_1, \ldots, p_{n_P}$, and the following equalities hold for $w = e_{i_1} \cdots e_{i_m}$, $m = 1, 2, \ldots$:*

$$A^w = P_{\mathcal{X}_{r(e_{i_1})}} \widetilde{A}^w|_{\mathcal{X}_{s(e_{i_m})}}, \tag{6.16}$$

$$(A\sharp B)^w = P_{\mathcal{X}_{r(e_{i_1})}} (\widetilde{A}\sharp\widetilde{B})^w, \tag{6.17}$$

$$(C\flat A)^w = (\widetilde{C}\flat\widetilde{A})^w|_{\mathcal{X}_{s(e_{i_m})}}, \tag{6.18}$$

$$(C\flat A\sharp B)^w = (\widetilde{C}\flat\widetilde{A}\sharp\widetilde{B})^w. \tag{6.19}$$

Proof. (i)⇒(iii). From the definition of dilation of non-commutative BGM systems it follows that $\widetilde{\mathcal{X}}_p \supset \mathcal{X}_p$ for each $p \in P$. By Remark 6.1 and Proposition 5.3, it follows from (5.5) that for $w = e_{i_1} \cdots e_{i_m}$, $m = 1, 2, \ldots$, and $k = 1, \ldots, N$ one has

$$\begin{aligned}
A^{we_k} &= A_{r(e_{i_1}),s(e_{i_2})} A_{r(e_{i_2}),s(e_{i_3})} \cdots A_{r(e_{i_{m-1}}),s(e_{i_m})} A_{r(e_{i_m}),s(e_k)} \\
&= P_{\mathcal{X}_{s(e_{i_1})}} I_{\Sigma,e_{i_1}} A I_{\Sigma,e_{i_2}} A \cdots I_{\Sigma,e_{i_m}} A|_{\mathcal{X}_{s(e_k)}} \\
&= P_{\mathcal{X}_{s(e_{i_1})}} \mathbf{A}^w|_{\mathcal{X}_{s(e_k)}} = P_{\mathcal{X}_{s(e_{i_1})}} \left(P_{\bigoplus_{s\in S} \mathcal{X}_s} \widetilde{\mathbf{A}}^w|_{\bigoplus_{s\in S} \mathcal{X}_s} \right) \Big|_{\mathcal{X}_{s(e_k)}} \\
&= P_{\mathcal{X}_{s(e_{i_1})}} \widetilde{\mathbf{A}}^w|_{\mathcal{X}_{s(e_k)}} = P_{\mathcal{X}_{s(e_{i_1})}} \left(P_{\widetilde{\mathcal{X}}_{s(e_{i_1})}} \widetilde{\mathbf{A}}^w|_{\widetilde{\mathcal{X}}_{s(e_k)}} \right) \Big|_{\mathcal{X}_{s(e_k)}} \\
&= P_{\mathcal{X}_{s(e_{i_1})}} \left(P_{\widetilde{\mathcal{X}}_{s(e_{i_1})}} I_{\widetilde{\Sigma},e_{i_1}} \widetilde{A} I_{\widetilde{\Sigma},e_{i_2}} \widetilde{A} \cdots I_{\widetilde{\Sigma},e_{i_m}} \widetilde{A}|_{\widetilde{\mathcal{X}}_{s(e_k)}} \right) \Big|_{\mathcal{X}_{s(e_k)}} \\
&= P_{\mathcal{X}_{s(e_{i_1})}} \widetilde{A}_{r(e_{i_1}),s(e_{i_2})} \widetilde{A}_{r(e_{i_2}),s(e_{i_3})} \cdots \widetilde{A}_{r(e_{i_{m-1}}),s(e_{i_m})} \widetilde{A}_{r(e_{i_m}),s(e_k)}|_{\mathcal{X}_{s(e_k)}} \\
&= P_{\mathcal{X}_{r(e_{i_1})}} \widetilde{A}^{we_k}|_{\mathcal{X}_{s(e_k)}}
\end{aligned}$$

(here we used the fact that $\mathcal{X}_{r(e_{i_1})} = \mathcal{X}_{s(e_{i_1})}$). Thus, (6.16) holds. The proof of equalities (6.17)–(6.19) is analogous.

(iii)⇒(ii). We define for $s \in S$:

$$\begin{aligned}
\mathcal{D}_s &:= \bigvee_{w=e_{i_1}\cdots e_{i_m}:\ m\ge 1,\, s(e_{i_1})=s} \widetilde{A}^w \left(\sum_{j=1}^{N} (\widetilde{A}_{r(e_{i_m}),s(e_j)} - A_{r(e_{i_m}),s(e_j)}) \mathcal{X}_{s(e_j)} \right. \\
&+ \left. (\widetilde{B}_{r(e_{i_m})} - B_{r(e_{i_m})}) \mathcal{U} \right).
\end{aligned}$$

Then $\mathcal{D}_s \subset \widetilde{\mathcal{X}}_{r(e_{i_1})} = \widetilde{\mathcal{X}}_{s(e_{i_1})} = \widetilde{\mathcal{X}}_s$. Moreover, $\mathcal{D}_s \perp \mathcal{X}_s$. Indeed, for arbitrary $j \in \{1, \ldots, N\}$, $x_j \in \mathcal{X}_{s(e_j)}$, $u \in \mathcal{U}$ and $w = e_{i_1} \cdots e_{i_m}$ with $m \ge 1$, $s(e_{i_1}) = s$, we have

$$\begin{aligned}
& P_{\mathcal{X}_s} \widetilde{A}^w \left(\widetilde{A}_{r(e_{i_m}),s(e_j)} x_j - A_{r(e_{i_m}),s(e_j)} x_j + \widetilde{B}_{r(e_{i_m})} u - B_{r(e_{i_m})} u \right) \\
&= P_{\mathcal{X}_s} \widetilde{A}^{we_j} x_j - (P_{\mathcal{X}_s} \widetilde{A}^w|_{\mathcal{X}_{s(e_{i_m})}}) \cdot A_{r(e_{i_m}),s(e_j)} x_j \\
&+ P_{\mathcal{X}_s} (\widetilde{A} \sharp \widetilde{B})^w u - (P_{\mathcal{X}_s} \widetilde{A}^w|_{\mathcal{X}_{s(e_{i_m})}}) \cdot B_{r(e_{i_m})} u \\
&= A^{we_j} x_j - A^w \cdot A_{r(e_{i_m}),s(e_j)} x_j + (A \sharp B)^w u - A^w \cdot B_{r(e_{i_m})} u \\
&= 0.
\end{aligned}$$

Hence $P_{\mathcal{X}_s} \mathcal{D}_s = \{0\}$, and $\mathcal{D}_s \perp \mathcal{X}_s$. Set

$$\mathcal{D}_{*,s} := \widetilde{X}_s \ominus (\mathcal{D}_s \oplus \mathcal{X}_s), \quad s \in S.$$

Then (6.10) is true. From the definition of $\mathcal{D}_s$ we obtain that $\widetilde{A}_{r(e),s} \mathcal{D}_s \subset \mathcal{D}_{s(e)}$, $s \in S$, $e \in E$. Further, for arbitrary $j \in \{1, \ldots, N\}$, $x_j \in \mathcal{X}_{s(e_j)}$, $u \in \mathcal{U}$ and $w =$

$e_{i_1}\cdots e_{i_m}$ with $m\geq 1$, $s(e_{i_1})=s$, we have

$$\begin{aligned}
&\widetilde{C_s}\widetilde{A}^w\left(\widetilde{A}_{r(e_{i_m}),s(e_j)}x_j - A_{r(e_{i_m}),s(e_j)}x_j + \widetilde{B}_{r(e_{i_m})}u - B_{r(e_{i_m})}u\right)\\
=\ & (\widetilde{C}\flat\widetilde{A})^{we_j}x_j - \left((\widetilde{C}\flat\widetilde{A})^w|_{\mathcal{X}_{s(e_{i_m})}}\right)\cdot A_{r(e_{i_m}),s(e_j)}x_j\\
+\ & (\widetilde{C}\flat\widetilde{A}\sharp\widetilde{B})^w u - \left((\widetilde{C}\flat\widetilde{A})^w|_{\mathcal{X}_{s(e_{i_m})}}\right)\cdot B_{r(e_{i_m})}u\\
=\ & (C\flat A)^{we_j}x_j - (C\flat A)^w\cdot A_{r(e_{i_m}),s(e_j)}x_j + (C\flat A\sharp B)^w u - (C\flat A)^w\cdot B_{r(e_{i_m})}u\\
=\ & 0.
\end{aligned}$$

From here we obtain that $\widetilde{C}_s\mathcal{D}_s=\{0\},\ s\in S$. For arbitrary $e\in E$, $s\in S$, $x_s\in\mathcal{X}_s$ we have

$$\widetilde{A}_{r(e),s}x_s = (\widetilde{A}_{r(e),s}x_s - A_{r(e),s}x_s) + A_{r(e),s}x_s \in \mathcal{D}_{s(e)}\oplus\mathcal{X}_{s(e)}.$$

It was shown above that $\widetilde{A}_{r(e),s}\mathcal{D}_s\subset\mathcal{D}_{s(e)}$. Hence, $\widetilde{A}_{r(e),s}(\mathcal{D}_s\oplus\mathcal{X}_s)\subset\mathcal{D}_{s(e)}\oplus\mathcal{X}_{s(e)}$. From here we get $(\widetilde{A}_{r(e),s})^*\mathcal{D}_{*,s(e)}\subset\mathcal{D}_{*,s}$. For an arbitrary $u\in\mathcal{U}$ we have

$$\widetilde{B}_{r(e)}u = (\widetilde{B}_{r(e)}u - B_{r(e)}u) + B_{r(e)}u \in \mathcal{D}_{s(e)}\oplus\mathcal{X}_{s(e)} = \widetilde{X}_{s(e)}\ominus\mathcal{D}_{*,s(e)}.$$

Then we get $(\widetilde{B}_{r(e)})^*\mathcal{D}_{*,s(e)}=\{0\},\ e\in E$. We have proved all the relations in (6.11). The equalities (6.12) are special cases of the equalities (6.16)–(6.18). Finally, the system $\widetilde{\Sigma}^{\mathrm{BGM,nc}}$ is a uniform dilation of the system $\Sigma^{\mathrm{BGM,nc}}$.

(ii)⇒(i) is obvious. □

Remark 6.4. It is easy to see that the dilation $\tilde{\Sigma}^{\mathrm{BGM,nc}}$ of a non-commutative BGM system $\Sigma^{\mathrm{BGM,nc}}$ is uniform if and only if it is uniform as a dilation in the sense of non-commutative FM systems (see Remark 6.1). Thus the equivalence (i)⇔(ii) follows also from Proposition 5.4.

In order to prove a conservative dilation theorem for non-commutative BGM systems we will need a unitary dilation lemma for 1D operator nodes. Let

$$U=\begin{bmatrix} A & B\\ C & D\end{bmatrix}\in\mathcal{L}(\mathcal{X}_1\oplus\mathcal{U},\mathcal{X}_2\oplus\mathcal{Y}).$$

We will call the collection of data $\alpha=(1;A,B,C,D;\mathcal{X}_1,\mathcal{X}_2,\mathcal{U},\mathcal{Y})$ a *1D contractive* (respectively, *unitary*) *operator node* if U is a contractive (respectively, unitary) operator. Clearly, in the case where $\mathcal{X}_1=\mathcal{X}_2\ (=\mathcal{X})$ we get the standard definition of 1D operator node, and in this case we write $\alpha=(1;A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y})$. The 1D node $\tilde{\alpha}=(1;\tilde{A},\tilde{B},\tilde{C},D;\tilde{\mathcal{X}}_1,\tilde{\mathcal{X}}_2,\mathcal{U},\mathcal{Y})$ is said to be a *dilation of the 1D node* $\alpha=(1;A,B,C,D;\mathcal{X}_1,\mathcal{X}_2,\mathcal{U},\mathcal{Y})$ if there exist subspaces $\mathcal{D}_1$ and $\mathcal{D}_{1,*}$ in $\tilde{\mathcal{X}}_1$, and subspaces $\mathcal{D}_2$ and $\mathcal{D}_{2,*}$ in $\tilde{\mathcal{X}}_2$ such that

$$\tilde{\mathcal{X}}_j=\mathcal{D}_j\oplus\mathcal{X}_j\oplus\mathcal{D}_{j,*},\quad j=1,2,$$

and with respect to these decompositions,

$$\tilde{A}=\begin{bmatrix} * & * & *\\ 0 & A & *\\ 0 & 0 & *\end{bmatrix},\quad \tilde{B}=\begin{bmatrix} *\\ B\\ 0\end{bmatrix},$$
$$\tilde{C}=\begin{bmatrix} 0 & C & *\end{bmatrix}.$$

Lemma 6.5. *Every contractive 1D node has a unitary dilation with* $\mathcal{D}_1=\mathcal{D}_2$, $\mathcal{D}_{1,*}=\mathcal{D}_{2,*}$.

Proof. For the case where $\mathcal{X}_1 = \mathcal{X}_2$ this Lemma is a well known fact [5].

Let $\alpha = (1; A, B, C, D; \mathcal{X}_1, \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be a contractive 1D node. Then so is $\beta = (1; 0, 0, 0, U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \{0\}, \mathcal{X}_1 \oplus \mathcal{U}, \mathcal{X}_2 \oplus \mathcal{Y})$. It follows from the paragraph above that there exists a unitary dilation $\hat{\beta} = (1; \hat{A}, \hat{B}, \hat{C}, U; \hat{\mathcal{X}}, \mathcal{X}_1 \oplus \mathcal{U}, \mathcal{X}_2 \oplus \mathcal{Y})$ of a node β, i.e., there exist Hilbert spaces $\mathcal{D}$ and $\mathcal{D}_*$ such that

$$\hat{\mathcal{X}} = \mathcal{D} \oplus \mathcal{D}_*,$$

and with respect to this decomposition and the decompositions $\mathcal{X}_1 \oplus \mathcal{U}$ and $\mathcal{X}_2 \oplus \mathcal{Y}$ of the input space and of the output space, respectively, of the node $\hat{\beta}$,

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ 0 & 0 \end{bmatrix},$$
$$\hat{C} = \begin{bmatrix} 0 & \hat{C}_{12} \\ 0 & \hat{C}_{22} \end{bmatrix},$$

and $\hat{U} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & U \end{bmatrix} \in \mathcal{L}(\hat{\mathcal{X}} \oplus (\mathcal{X}_1 \oplus \mathcal{U}), \hat{X} \oplus (\mathcal{X}_2 \oplus \mathcal{Y}))$ is a unitary operator. Define a node $\tilde{\alpha} = (1; \tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \mathcal{U}, \mathcal{Y})$, where

$$\tilde{\mathcal{X}}_1 = \mathcal{D} \oplus \mathcal{X}_1 \oplus \mathcal{D}_*, \quad \tilde{\mathcal{X}}_2 = \mathcal{D} \oplus \mathcal{X}_2 \oplus \mathcal{D}_*,$$

and with respect to these decompositions,

$$\tilde{A} = \begin{bmatrix} \hat{A}_{11} & \hat{B}_{11} & \hat{A}_{12} \\ 0 & A & \hat{C}_{12} \\ 0 & 0 & \hat{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \hat{B}_{12} \\ B \\ 0 \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} 0 & C & \hat{C}_{22} \end{bmatrix}.$$

Clearly, $\tilde{\alpha}$ is a dilation of α. Since the operator

$$\tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{bmatrix} \in \mathcal{L}((\mathcal{D} \oplus \mathcal{X}_1 \oplus \mathcal{D}_*) \oplus \mathcal{U}, (\mathcal{D} \oplus \mathcal{X}_2 \oplus \mathcal{D}_*) \oplus \mathcal{Y})$$

is obtained from the operator

$$\hat{U} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & U \end{bmatrix} \in \mathcal{L}((\mathcal{D} \oplus \mathcal{D}_*) \oplus (\mathcal{X}_1 \oplus \mathcal{U}), (\mathcal{D} \oplus \mathcal{D}_*) \oplus (\mathcal{X}_2 \oplus \mathcal{Y}))$$

by the permutation of subspaces:

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \hat{U} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$\tilde{U}$ is a unitary operator. It follows that the 1D node $\tilde{\alpha}$ is a unitary dilation of the 1D node α. □

Theorem 6.6. *Every dissipative non-commutative BGM system has a (uniform) conservative dilation.*

Proof. Let the non-commutative BGM system $\Sigma^{\mathrm{BGM,nc}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be dissipative. Then the operator

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}\left(\left(\bigoplus_{s\in S}\mathcal{X}_s\right) \oplus \mathcal{U}, \left(\bigoplus_{r\in R}\mathcal{X}_r\right) \oplus \mathcal{Y}\right)$$

is contractive, i.e., the 1D node $\alpha = (1; A, B, C, D; \bigoplus_{s\in S}\mathcal{X}_s, \bigoplus_{r\in R}\mathcal{X}_r, \mathcal{U}, \mathcal{Y})$ is contractive. By Proposition 6.5, this 1D node α has a unitary dilation $\breve{\alpha} = (1; \breve{A}, \breve{B}, \breve{C}, D; \breve{\mathcal{X}}_1, \breve{\mathcal{X}}_2, \mathcal{U}, \mathcal{Y})$, where

$$\breve{\mathcal{X}}_1 = \mathcal{D} \oplus \left(\bigoplus_{s\in S}\mathcal{X}_s\right) \oplus \mathcal{D}_*, \quad \breve{\mathcal{X}}_2 = \mathcal{D} \oplus \left(\bigoplus_{r\in R}\mathcal{X}_r\right) \oplus \mathcal{D}_*.$$

Define the 1D node $\hat{\alpha} = (1; \hat{A}, \hat{B}, \hat{C}, D; \hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2, \mathcal{U}, \mathcal{Y})$, where

$$\begin{aligned}
\hat{\mathcal{X}}_1 &= \left(\bigoplus_{k=-\infty}^{-1}\mathcal{D}^{(k)}\right) \oplus \left(\bigoplus_{s\in S}\mathcal{X}_s\right) \oplus \left(\bigoplus_{k=1}^{\infty}\mathcal{D}_*^{(k)}\right) \\
&= \left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_1 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right), \\
\hat{\mathcal{X}}_2 &= \left(\bigoplus_{k=-\infty}^{-1}\mathcal{D}^{(k)}\right) \oplus \left(\bigoplus_{r\in R}\mathcal{X}_r\right) \oplus \left(\bigoplus_{k=1}^{\infty}\mathcal{D}_*^{(k)}\right) \\
&= \left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_2 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right),
\end{aligned}$$

$\mathcal{D}^{(k)}$ (respectively, $\mathcal{D}_*^{(k)}$) are different copies of $\mathcal{D}$ (respectively, $\mathcal{D}_*$), $k = 1, 2, \ldots$, and

$$\begin{aligned}
\hat{A} &= \begin{bmatrix} I & 0 & 0 \\ 0 & \breve{A} & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathcal{L}\left(\left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_1 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right),\right. \\
&\qquad \left.\left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_2 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right)\right), \\
\hat{B} &= \begin{bmatrix} 0 \\ \breve{B} \\ 0 \end{bmatrix} \in \mathcal{L}\left(\mathcal{U}, \left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_2 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right)\right), \\
\hat{C} &= \begin{bmatrix} 0 & \breve{C} & 0 \end{bmatrix} \in \mathcal{L}\left(\left(\bigoplus_{k=-\infty}^{-2}\mathcal{D}^{(k)}\right) \oplus \breve{\mathcal{X}}_1 \oplus \left(\bigoplus_{k=2}^{\infty}\mathcal{D}_*^{(k)}\right), \mathcal{Y}\right).
\end{aligned}$$

Clearly, $\hat{\alpha}$ is a unitary dilation of $\breve{\alpha}$. Since, in turn, $\breve{\alpha}$ is a unitary dilation of α, so is $\hat{\alpha}$.

Let $S_1 = \{s_1, \ldots, s_\nu\}$ (respectively, $R_1 = \{r_1, \ldots, r_\mu\}$) be the first path component in the set S (respectively, R) of source (respectively, range) vertices of the graph G. Set

$$\begin{aligned}
\mathcal{D}_{s_i} &= \bigoplus_{j=-\infty}^{0} \mathcal{D}^{(-i+j\nu)}, \quad \mathcal{D}_{*,s_i} = \bigoplus_{j=0}^{\infty} \mathcal{D}_*^{(i+j\nu)}, \quad i=1,\ldots,\nu,\\
\mathcal{D}_{s_i} &= \{0\} = \mathcal{D}_{*,s_i}, \quad i=\nu+1,\ldots,n_S,\\
\mathcal{D}_{r_l} &= \bigoplus_{j=-\infty}^{0} \mathcal{D}^{(-l+j\mu)}, \quad \mathcal{D}_{*,r_l} = \bigoplus_{j=0}^{\infty} \mathcal{D}_*^{(l+j\mu)}, \quad l=1,\ldots,\mu,\\
\mathcal{D}_{r_l} &= \{0\} = \mathcal{D}_{*,r_l}, \quad l=\mu+1,\ldots,n_R,\\
\tilde{\mathcal{X}}_{s_i} &= \mathcal{D}_{s_i} \oplus \mathcal{X}_{s_i} \oplus \mathcal{D}_{*,s_i}, \quad i=1,\ldots,n_S,\\
\tilde{\mathcal{X}}_{r_l} &= \mathcal{D}_{r_l} \oplus \mathcal{X}_{r_l} \oplus \mathcal{D}_{*,r_l}, \quad l=1,\ldots,n_R.
\end{aligned}$$

Then the spaces $\tilde{\mathcal{X}}_1 = \bigoplus_{i=1}^{n_S} \tilde{\mathcal{X}}_{s_i}$ and $\tilde{\mathcal{X}}_2 = \bigoplus_{l=1}^{n_R} \tilde{\mathcal{X}}_{r_l}$ are obtained from the spaces $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$, respectively, by subspace rearrangement transformations $T_1 = T_1^* = T_1^{-1}$ and $T_2 = T_2^* = T_2^{-1}$. The corresponding 1D node $\tilde{\alpha} = (1; \tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \mathcal{U}, \mathcal{Y})$ is *unitarily similar* to the 1D node $\hat{\alpha} = (1; \hat{A}, \hat{B}, \hat{C}, D; \hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2, \mathcal{U}, \mathcal{Y})$, i.e., with unitary operators $T_1 \in \mathcal{L}(\hat{\mathcal{X}}_1, \tilde{\mathcal{X}}_1)$ and $T_2 \in \mathcal{L}(\hat{\mathcal{X}}_2, \tilde{\mathcal{X}}_2)$,

$$\tilde{A} = T_2 \hat{A} T_1^{-1}, \quad \tilde{B} = T_2 \hat{B}, \quad \tilde{C} = \hat{C} T_1^{-1}.$$

Since

$$\begin{aligned}
\tilde{U} &= \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{bmatrix} = \begin{bmatrix} T_2 & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}\\
&= \begin{bmatrix} T_2 & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \hat{U} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}
\end{aligned}$$

is a unitary operator, $\tilde{\alpha}$ is a unitary 1D node. Then the system $\tilde{\Sigma}^{\rm BGM,nc} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, with $\tilde{\mathcal{X}} = \{\tilde{\mathcal{X}}_p\}_{p \in P}$, is conservative. It is now straightforward to check that the relations (6.1)–(6.3) hold, and thus $\tilde{\Sigma}^{\rm BGM,nc}$ is a conservative dilation of $\Sigma^{\rm BGM,nc}$. □

Remark 6.7. For the case where $\mu = \nu$ it suffices to have finitely many copies of the spaces $\mathcal{D}$ and $\mathcal{D}_*$:

$$\begin{aligned}
\hat{\mathcal{X}}_1 &= \left(\bigoplus_{k=-\nu}^{-1} \mathcal{D}^{(k)} \right) \oplus \left(\bigoplus_{s \in S} \mathcal{X}_s \right) \oplus \left(\bigoplus_{k=1}^{\nu} \mathcal{D}_*^{(k)} \right),\\
\hat{\mathcal{X}}_2 &= \left(\bigoplus_{k=-\nu}^{-1} \mathcal{D}^{(k)} \right) \oplus \left(\bigoplus_{r \in R} \mathcal{X}_r \right) \oplus \left(\bigoplus_{k=1}^{\nu} \mathcal{D}_*^{(k)} \right),
\end{aligned}$$

then define

$$\begin{aligned}
\mathcal{D}_{s_i} &= \mathcal{D}_{r_i} = \mathcal{D}^{(-i)}, \quad \mathcal{D}_{*,s_i} = \mathcal{D}_{*,r_i} = \mathcal{D}_*^{(i)}, \quad i=1,\ldots,\nu,\\
\mathcal{D}_{s_i} &= \{0\} = \mathcal{D}_{*,s_i}, \quad i=\nu+1,\ldots,n_S,\\
\mathcal{D}_{r_l} &= \{0\} = \mathcal{D}_{*,r_l}, \quad l=\mu+1,\ldots,n_R,\\
\tilde{\mathcal{X}}_{s_i} &= \mathcal{D}_{s_i} \oplus \mathcal{X}_{s_i} \oplus \mathcal{D}_{*,s_i}, \quad i=1,\ldots,n_S,\\
\tilde{\mathcal{X}}_{r_l} &= \mathcal{D}_{r_l} \oplus \mathcal{X}_{r_l} \oplus \mathcal{D}_{*,r_l}, \quad l=1,\ldots,n_R,
\end{aligned}$$

and proceed in the same way as in the proof of Theorem 6.6 for the definitions of $\hat{\alpha}, \tilde{\alpha}$, and $\tilde{\Sigma}^{\rm BGM,nc}$.

6.2. **The commutative setting.** We will say that the commutative BGM system $\tilde{\Sigma}^{\mathrm{BGM,c}} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a *dilation of the commutative BGM system* $\Sigma^{\mathrm{BGM,c}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ if there exist subspaces $\mathcal{D}_s$ and $\mathcal{D}_{*,s}$ in $\tilde{\mathcal{X}}_s$, $s \in S$, such that (6.10)–(6.12) hold, i.e., if the associated non-commutative BGM system $\tilde{\Sigma}^{\mathrm{BGM,nc}} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a uniform dilation of the associated non-commutative BGM system $\Sigma^{\mathrm{BGM,nc}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. Let us note that the structure of BGM systems forces this definition of dilation for commutative BGM systems to be the most natural. E.g., if one tries to adapt the definition of dilation of a non-commutative BGM system from Section 6.1 to the commutative case by considering subspaces $\mathcal{D}_{z,s}$ and $\mathcal{D}_{*,z,s}$ for $z \in \mathbb{C}^N$ satisfying (6.1) and (6.2) then these subspaces turn out to be independent of the value of z, thus the dilation can be chosen uniform.

Proposition 6.8. *The transfer functions of a system* $\Sigma^{\mathrm{BGM,c}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *and of its dilation* $\tilde{\Sigma}^{\mathrm{BGM,c}} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ *coincide.*

Proof. By Proposition 6.2, the transfer functions of the associated non-commutative BGM systems $\Sigma^{\mathrm{BGM,nc}} = (G; U; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\tilde{\Sigma}^{\mathrm{BGM,nc}} = (G; \tilde{U}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ coincide: $T_{\Sigma^{\mathrm{BGM,nc}}} = T_{\tilde{\Sigma}^{\mathrm{BGM,nc}}}$. Replacing the non-commuting indeterminates by the commuting ones gives the equality $T_{\Sigma^{\mathrm{BGM,c}}} = T_{\tilde{\Sigma}^{\mathrm{BGM,c}}}$. □

From Theorem 6.6 and the definition of dilation of a commutative BGM system we obtain the following.

Theorem 6.9. *Every dissipative commutative BGM system has a conservative dilation.*

Thus, we see some discrepancy in properties of dissipative commutative BGM systems, on the one hand, and of dissipative commutative KV systems and $\mathbb{D}^N$-dissipative FM systems, on the other hand.

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Department of Mathematics, Virginia Tech, Blacksburg, VA, 24061
E-mail address: ball@math.vt.edu

Department of Mathematics, Drexel University, 3141 Chestnut Str., Philadelphia, PA, 19104
E-mail address: dmitryk@math.drexel.edu